Square-Triangle-Rhombus Random Tiling

Maria Tsarenko

in collaboration with Jan de Gier

Department of Mathematics and Statistics
The University of Melbourne

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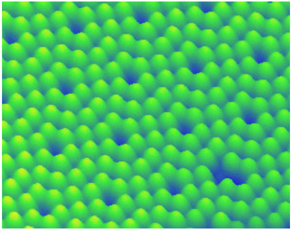
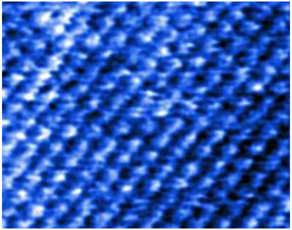
Crystals

Crystals are invariant under discrete translations and rotations.

Atomic structure can be modelled with periodic lattices.
Translational symmetry and 2-, 3-, 4-, 6-fold rotational symmetries.



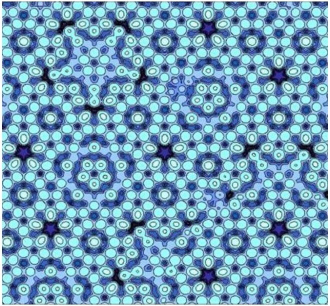
Silicon (Si)
– hexagonal lattice



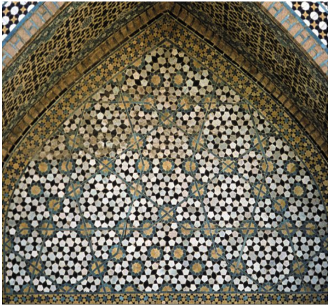
Sodium Chloride (NaCl)
– square lattice

Quasi-periodicity in Nature

Both exhibit 5-fold rotational symmetry but no translational:



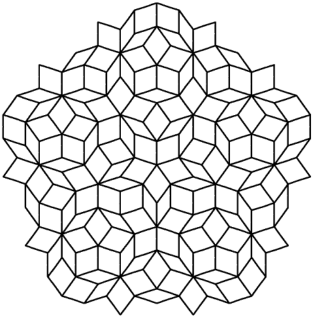
Atomic model of Al-Pd-Mn quasicrystal surface



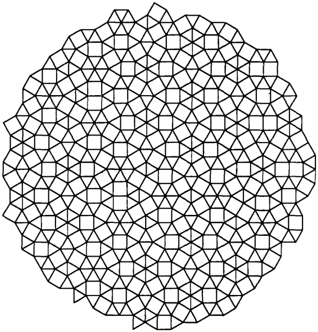
Mosaic on the wall of Darb-e Imam Shrine in Isfahan, Iran, circ. 1450

Quasi-periodic Tilings

possess no translational symmetry, only ‘forbidden’ rotational symmetry



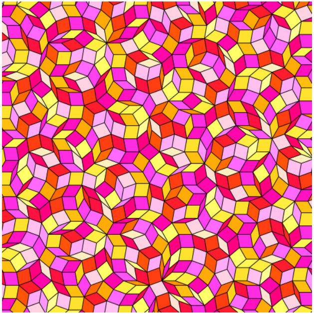
Penrose tiling with rhombi
5-fold symmetry



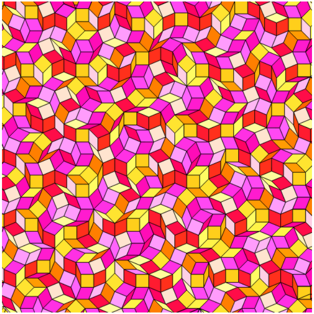
Square-Triangle tiling
12-fold symmetry

Quasi-periodic and Random Tilings

Quasi-periodic tiling – projection of a cut of a n-dimensional cube by a plane with an irrational slope.
Random tiling – projection of a cut of a n-dimensional cube by a surface.
n= 9:



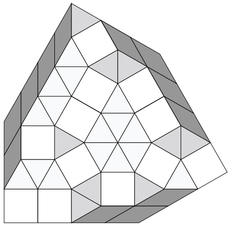
Quasi-periodic tiling

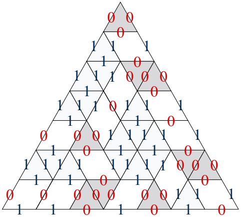


Random tiling

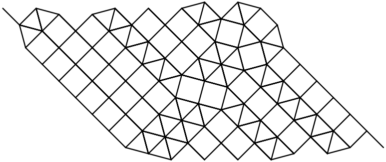
- *Random tilings* are entropic models for quasicrystals. They do not require strict matchings and still exhibit ‘forbidden’ rotational symmetries in a statistical sense.

- *Random tilings* are entropic models for quasicrystals. They do not require strict matchings and still exhibit ‘forbidden’ rotational symmetries in a statistical sense.
- Square-Triangle tiling was recently found to be related to puzzles of Knutson–Tao–Woodward, which compute Littlewood–Richardson coefficients.

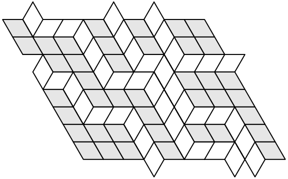




Square-Triangle-Rhombus Tiling Model



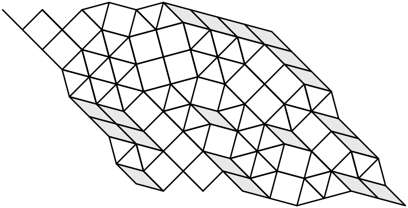
Square-Triangle tiling



Rhombus tiling

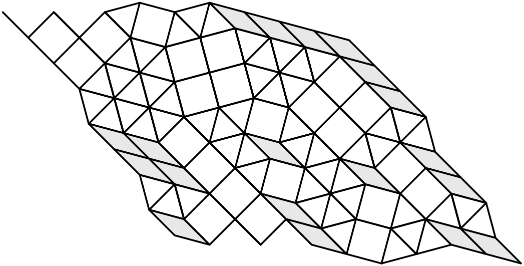
Exact results for the square-triangle tiling model were obtained by Widom and Kalugin. Solved using the algebraic Bethe Ansatz by de Gier and Nienhuis.

In unpublished work de Gier and Nienhuis proposed an extension of square-triangle tiling with a new 'thin rhombus' tile:



Formulating the problem

New model consists of filling a region of the plane with squares, triangles and thin rhombi in restricted orientations as shown:



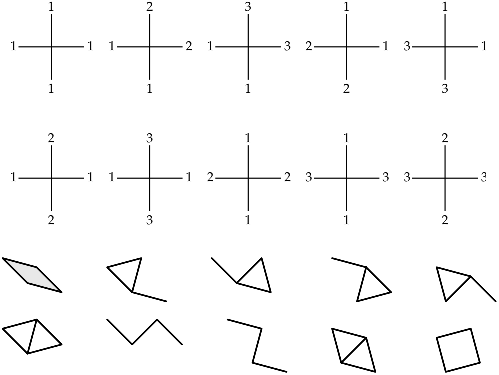
weight() $= u$, weight()$=e^{\mu}$, weight(all other tiles) $=1$.

Want to compute entropy σ as a function of u and μ

$$Z_A = \sum_{\text{config}} \text{weight(config)},$$

$$\sigma(u,\mu) = \lim_{A\to\infty} \frac{\log Z}{A} - \log u \; n$$$$- \mu \; n$$

Correspondence between the 10-Vertex Model and the Square-Triangle-Rhombus Tiling.

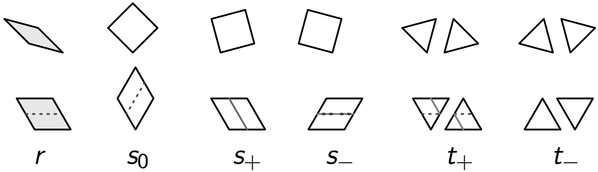


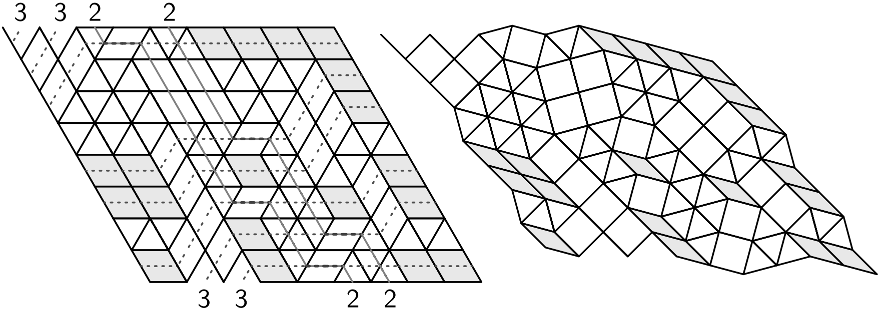
10-Vertex Model

S-T-R tiling

Cruicially, this lattice model is *integrable*. Can use the *transfer matrix* method and the method of algebraic *Bethe Ansatz* to compute σ.

Deformed $M \times N$ lattice with p.b.c. and associated propagation of two types of particles: '2's and '3's.





- The eigenvalue Λ of the transfer matrix is given by:

$$\Lambda(u) = u^N \prod_{i=1}^{M_1} \frac{1}{u - u_i} + \mathrm{e}^{-M_2\mu} \prod_{i=1}^{M_1} \frac{1}{u_i - u} \prod_{j=1}^{M_2} \frac{1}{u - v_j} + \\ + \mathrm{e}^{(M_1 - M_2)\mu} \prod_{j=1}^{M_2} \frac{1}{v_j - u}.$$

Two types of particles / Bethe Ansatz root types:
M_1 of u_i and M_2 of v_j.

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  Two types of particles / Bethe Ansatz root types:
  M_1 of u_i and M_2 of v_j.

- The Bethe Ansatz equations is a coupled system of non-linear equations:

$$u_i^N + (-)^{M_1} \prod_{j=1}^{M_2} \frac{\mathrm{e}^{-\mu}}{u_i - v_j} = 0,$$

$$1 + (-)^{M_2} \prod_{i=1}^{M_1} \frac{\mathrm{e}^{-\mu}}{u_i - v_j} = 0.$$

Taking the logarithm of both sides, the BA equations can be rewritten as

$$\begin{aligned} N\ln(u_i) + \sum_{j=1}^{M_2} \ln(u - v_j) - (M_1 - 1)i\pi + \mu M_2 \; &= 2\pi i I_i, \qquad I_i \in \mathbb{Z}, \\ \sum_{i=1}^{M_1} \ln(u_i - v) - (M_2 - 1)i\pi + \mu M_1 \; &= 2\pi i \tilde{I}_j, \qquad \tilde{I}_j \in \mathbb{Z} \end{aligned}$$

Define $F_1(u)$ and $F_2(v)$:

$$\begin{aligned} F_1(u) \; &= \; \ln(u) + \frac{1}{N}\sum_{j=1}^{M_2} \ln(u - v_j) + m_2\mu, \quad m_2 = \frac{M_2}{N}, \\ F_2(v) \; &= \; \frac{1}{N}\sum_{i=1}^{M_1} \ln(u_i - v) + m_1\mu, \qquad\qquad m_1 = \frac{M_1}{N} \end{aligned}$$

Then the BA equations and Λ can be rewritten in terms of F_1 and F_2:

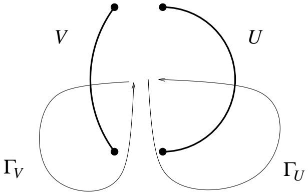
$$\mathrm{Re}\,[F_1(u_i)] = \mathrm{Re}\,[F_2(v_j)] = 0,$$

$$\Lambda(u) = u^N \mathrm{e}^{N m_1 \mu} \left(\mathrm{e}^{-NF_1(u)} + \mathrm{e}^{-NF_1(u) - NF_2(u)} + \mathrm{e}^{-NF_2(u)} \right).$$

Differentiating F_1 and F_2 and taking the thermodynamic limit ($N \to \infty$) the root density functions f_1 and f_2 are given by a system of integral equations:

$$f_1(u) = \frac{1}{u} - \frac{1}{2\pi i} \int_V \frac{f_2(v)}{v-u}\, dv$$
$$f_2(v) = -\frac{1}{2\pi i} \int_U \frac{f_1(u)}{u-v}\, du.$$

f_1 is analytic on $\mathbb{C} \backslash V \cup \{0\}$ and f_2 is analytic on $\mathbb{C} \backslash U$.



We are interested in the analytic continuation of f_1 across the cut V, moving along the path Γ_V, and, similarly, in the analytic continuation of f_2 across the cut U, moving along the path Γ_U.

For a general linear combination,

$$G(z) = a_1 f_1(z) + a_2 f_2(z),$$

If endpoints of V and U coincide, then closed path Γ necessarily crosses both V and U.

The analytic continuations of $G(z)$ along $\Gamma = \Gamma_U \Gamma_V$ is given by the monodromy matrix:

$$\Gamma : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

and we observe that the monodromy group is isomorphic to $\mathbb{Z}_6$:

$$(\Gamma_U \Gamma_V)^6 = \mathrm{I}$$

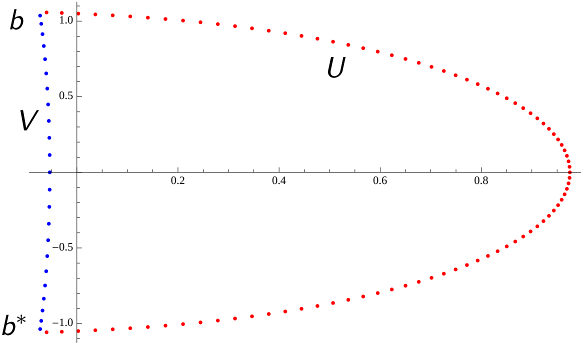


Figure : Bethe Ansatz roots (for $N = 152$, $M_1 = 160$, $M_2 = 39$, and $\gamma = -0.07$). The roots $\{v_j\}$ and $\{u_i\}$ lie on a curves V and U respectively, in the complex plane.

f_1 and f_2 are made single-valued functions with the change of variables:

$$t = \left(\frac{zb^{-1} - 1}{1 - zb^{*-1}} \right)^{1/6}, \quad z = b\frac{1 + t^6}{1 + bb^{*-1}t^6},$$

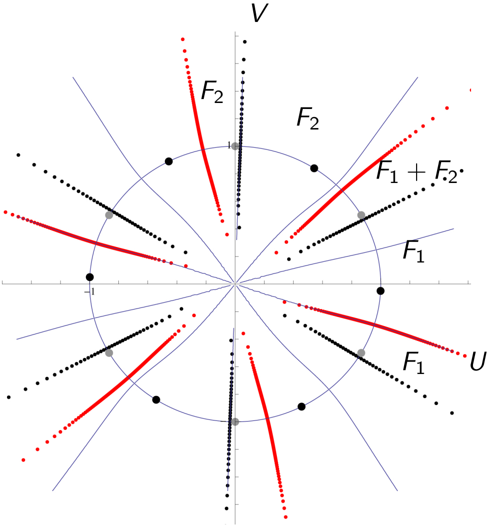
$$\begin{aligned} z = b \quad &\mapsto \quad t = 0 \\ z = b^* \quad &\mapsto \quad t \to \infty \end{aligned}$$

In the original variable z, f_1 and f_2 correspond to different sheets of the Riemann surface of the same function $G(z)$.

b is parametrised as follows:

$$b = i|b|e^{-i\gamma}, \quad \gamma \in (-\pi/2, \pi/2).$$

$G(z)dz$



Images of the Bethe Ansatz roots in the t-plane.

The t-plane is divided into 12 sectors.

Dots placed at points $z = 0$ and $z = \infty$, with $z = 0$ lying on the lines corresponding to the images of the V-cut.

Reconstructing $G(z)\mathrm{d}z$ from its singularities

$$G(z)\mathrm{d}z = \sum_{k=1}^{12} \frac{r_k}{t - t_k}\mathrm{d}t.$$

We find

$$G(z) = \frac{1}{z}\left(\frac{t + t^{-1}}{\sqrt{3}} - C(t + t^{-5}) - C^*(t^{-1} + t^5)\right),$$
$$C = \frac{1}{2\sqrt{3}\cos\gamma}\left(\mathrm{e}^{\mathrm{i}\gamma} + \mathrm{e}^{2\mathrm{i}\gamma/3}(\mathrm{i}m_1 - \mathrm{e}^{\mathrm{i}\pi/6}(1 + m_2))\right).$$

The images of U and V in the t-plane meet at $t = 0$ and $t = \infty$, which implies that the $G(z)\mathrm{d}z$ vanishes there and that $C = 0$. This fixes m_1 and m_2 to be:

$$m_1^* = \frac{2}{\sqrt{3}}\cos\frac{\pi + \gamma}{3},$$
$$m_2^* = \frac{2}{\sqrt{3}}\cos\frac{\gamma}{3} - 1.$$

The eigenvalue Λ can be calculated by considering the following integral,

$$I(\theta, \gamma) = \mathrm{Re} \int_0^{\mathrm{e}^{\mathrm{i}\theta}} \frac{t + t^{-1}}{z\sqrt{3}} \frac{\mathrm{d}z}{\mathrm{d}t} \mathrm{d}t.$$

$F_i(u)$ are given by $I(\theta, \gamma)$ for appropriate intervals of θ, where

$$u(\theta, \gamma) = |b(\gamma)| \frac{\cos 3\theta}{\sin(\gamma - 3\theta)}.$$

Recall that Λ was given in terms of F_is:

$$\frac{1}{N}\log\Lambda(u) = m_1\mu + \log(u) - \min \mathrm{Re}\left\{F_1(u), F_1(u)+F_2(u), F_2(u)\right\}$$

We find the largest eigenvalue

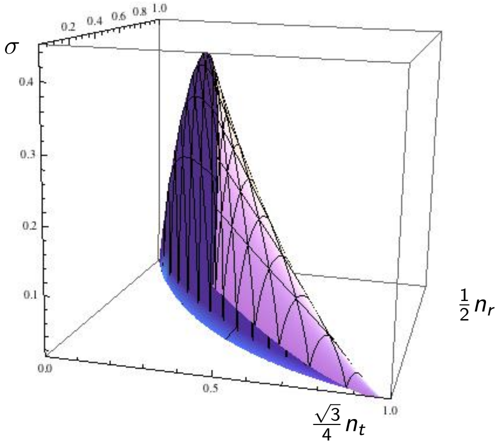
$$\frac{1}{N}\log\Lambda_{\max}(u) = m_1\mu + \log(u) - I\left(\theta,\gamma\right),$$
$$\text{with } \theta \in \left(\frac{\pi}{6},\frac{\pi}{2}\right) \text{ and } \gamma \in \left(0,\frac{\pi}{2}\right)$$

Using the Legendre transformation, we were able to compute the entropy in terms of tile tensities: n_r and n_{s_+}

$$\sigma(n^*_{s_+}, n^*_r) = \mu(m_1 - n^*_{s_+}) + (1-n^*_r)\log(u) - I\left(\theta,\gamma\right).$$

Bulk entropy σ

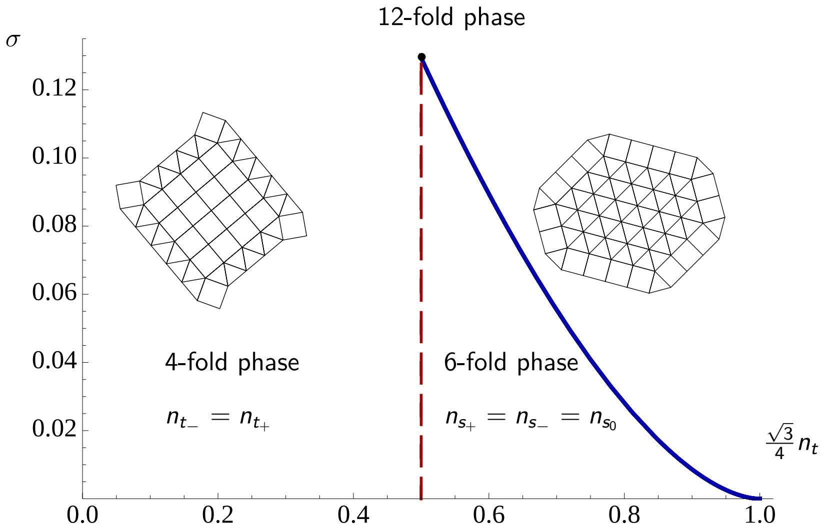
Can solve for n_r and n_{s_+} in terms of u and $\mu(\gamma)$, and for all other tile densities in terms of n_r, n_{s_+} and BA root densities m_1, m_2.



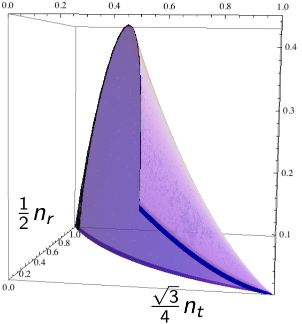
Bulk entropy σ as a function of the total triangle area fraction $\frac{\sqrt{3}}{4} n_t$

Previously known results without rhombi, ie. $u = 0$

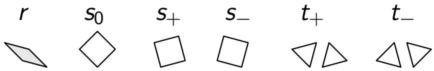
Bulk entropy σ as a function of the total triangle area fraction $\frac{\sqrt{3}}{4} n_t$



Line of maximum entropy and symmetry



$$n_{s_-} = n_{s_0}$$
$$n_{t_-} = n_{t_+}$$



Densities on the line of maximum entropy in terms of γ are:

$$n_r = \frac{4}{\sqrt{3}} \sin\left[\frac{\pi}{6} + \frac{\gamma}{3}\right] \sin\left[\frac{\gamma}{3}\right]$$

$$n_{t_-} = 2 - \frac{4}{\sqrt{3}} \cos\left[\frac{1}{6}(\pi + 2\gamma)\right] + 4\cos\left[\frac{1}{6}(\pi + 4\gamma)\right]$$

$$n_{s_+} = \frac{1}{\sqrt{3}}\left(4\cos\left[\frac{\pi+\gamma}{3}\right] - 2\cos\left[\frac{1}{6}(\pi + 4\gamma)\right]\right)$$

$$n_{s_0} = m_2 = \frac{2}{\sqrt{3}} \cos\frac{\gamma}{3} - 1.$$

σ_{max}

Expression for the line of maximum entropy in terms of γ:

$$
\begin{aligned}
&\sigma_{\text{max line}}(\gamma) = \\
&= \cos\left[\frac{1}{6}(\pi+4\gamma)\right]\left(\frac{\log(432)+2\log\left(\tan\left[\frac{1}{12}(\pi-2\gamma)\right]\tan[\gamma]\tan\left[\frac{1}{12}(\pi+2\gamma)\right]\right.}{\sqrt{3}}\right. \\
&\qquad +\log\left(\tan\left[\frac{\pi}{4}+\frac{\gamma}{6}\right]\tan\left[\frac{1}{12}(\pi+2\gamma)\right]\right)\sec\left[\frac{\pi+\gamma}{3}\right]\Bigg) - \\
&\qquad -\frac{2}{\sqrt{3}}\cos\left[\frac{\pi+\gamma}{3}\right]\left(\log\left(\tan\left[\frac{\pi}{4}+\frac{\gamma}{6}\right]\tan\left[\frac{1}{12}(\pi+2\gamma)\right]\right)+\right. \\
&\qquad +2\log\left(\tan\left[\frac{1}{12}(\pi-2\gamma)\right]\tan\left[\frac{1}{12}(\pi+2\gamma)\right]\right)\left(\sin\left[\frac{\pi+\gamma}{3}\right]-1\right)\Bigg) - \\
&\qquad \log\left(\tan\left[\frac{\pi}{6}+\frac{\gamma}{3}\right]\tan\left[\frac{\gamma}{3}\right]\right), \text{ where } \gamma\in\left(0,\frac{\pi}{2}\right)
\end{aligned}
$$

Conclusion

- We solved the model on a two-dimensional surface, where the densities and all other quantities are parametrised in terms of u and μ.
- We computed an explicit expression for the line of maximum entropy, which is also the line of maximum symmetry.
- Further directions: solving for general U and V; the extension of these methods to other integrable tiling models: octagonal and decagonal triangle-rectangle tiling models; possible connections to combinatorics.