Solvable Off-critical Logarithmic Minimal Models

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Joint work with Paul A. Pearce

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Outline

- The Ising model of a magnet as a rational CFT with a finite operator content.
- Yang-Baxter integrability and Corner Transfer Matrices (CTMs).
- RSOS lattice models and minimal models $\mathcal{M}(m,m')$ as simplest rational CFTs.
- The *logarithmic limit* $$\lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \mathcal{M}(m,m') = \mathcal{LM}(p,p')$$
- What we did!
- RSOS Generalized Order Parameters (GOPs) and their associated critical exponents and conformal weights.
- Off-critical solution of the logarithmic minimal models. Logarithmic limit of RSOS GOPs, associated critical exponents and conformal weights.

Exact Solution of the Ising Model

- The Ising model was solved exactly in 1944 by Onsager for the limiting free energy f

The specific heat $f''(T)$ diverges logarithmically at $T = T_c$ with a *critical exponent* $\alpha = 0$

$$f(T) \sim (T-T_c)^{2-\alpha}, \qquad T-T_c \to 0, \qquad f''(T) \sim \log(T-T_c), \qquad \alpha = 0_{\log}$$

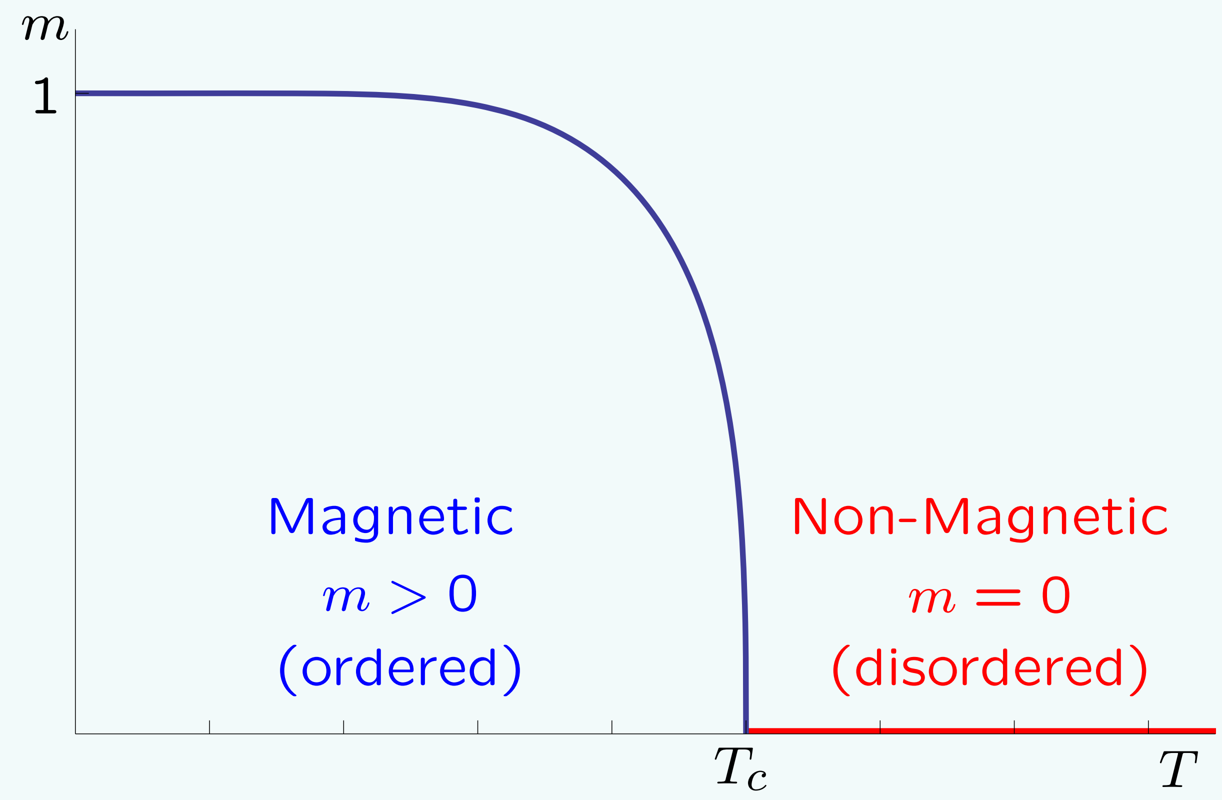
- The magnetization of the Ising model was calculated exactly in 1949 by C.N. Yang

$$m = \langle \sigma \rangle_+ = \lim_{N\to\infty} \frac{\sum\limits_{\text{spins}} \prod\limits_{\text{faces}} \sigma\, W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)}{\sum\limits_{\text{spins}} \prod\limits_{\text{faces}} W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)}$$

The magnetization vanishes above T_c with spontaneous magnetization below T_c

$$m \sim (T_c - T)^{\beta}, \qquad T - T_c \to 0-, \qquad \beta = 1/8$$

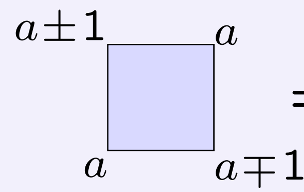
This One Point Function (OPF) is an example of an *order parameter*.

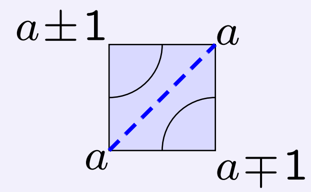


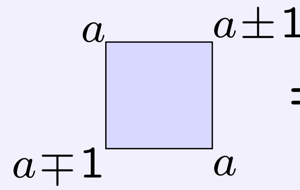
- Critical exponents such as α, β are *universal* (independent of the anisotropy or lattice structure) and described by a Conformal Field Theory (CFT) in the continuum scaling limit.
- The ABF model with 3 states (+, “frozen”, −) realizes the Ising model.

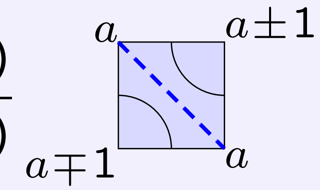
RSOS Models

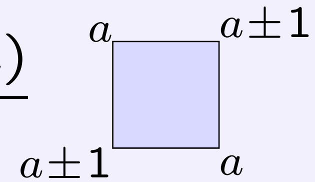
- The statistical face weights of the Restricted Solid-On-Solid (RSOS) models (Andrews-Baxter-Forrester 1984, Forrester-Baxter 1985) are

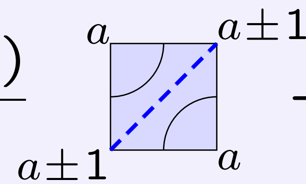
$$W\left(\begin{matrix} a\pm 1 & a \\ a & a\mp 1\end{matrix}\,\middle|\, u\right) = \frac{s(\lambda-u)}{s(\lambda)}$$


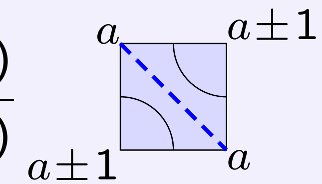
$$=\frac{s(\lambda-u)}{s(\lambda)}$$


$$W\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a\end{matrix}\,\middle|\, u\right) = \frac{s((a\pm 1)\lambda)}{s(a\lambda)}\frac{s(u)}{s(\lambda)}$$


$$=\frac{s((a\pm 1)\lambda)}{s(a\lambda)}\frac{s(u)}{s(\lambda)}$$


$$W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\end{matrix}\,\middle|\, u\right) = \frac{s(a\lambda\pm u)}{s(a\lambda)}$$


$$=\frac{c(0)c((a\pm 1)\lambda\pm u)}{c((a\pm 1)\lambda)c(u)}\frac{s(\lambda-u)}{s(\lambda)}$$


$$+\frac{c(\lambda)c(a\lambda\pm u)}{c((a\pm 1)\lambda)c(u)}\frac{s((a\pm 1)\lambda)}{s(a\lambda)}\frac{s(u)}{s(\lambda)}$$


The decomposition into decorated tiles allows to describe the nonlocal statistics of height clusters and loop connectivities (percolation properties).

- Here $s(u)=\vartheta_1(u,t)$, $c(u)=\vartheta_4(u,t)$ are elliptic theta functions, with nome t.

At criticality, $t=0$. $t\to 1$ is the ordered limit.

- u relates to spatial anisotropy, and the model-dependent *crossing parameter* λ is

$$\lambda=\frac{(m'-m)\pi}{m'},\qquad 2\le m<m',\qquad m,m' \text{ coprime}$$

Yang-Baxter Integrability

- A 2-d lattice model is exactly solvable if the face weights satisfy the Yang-Baxter equation.
- The RSOS models satisfy YBE and are integrable.
- YBE implies commuting row and Corner Transfer Matrices (CTMs) and hence integrability.

- OPFs are calculated using Baxter's CTMs

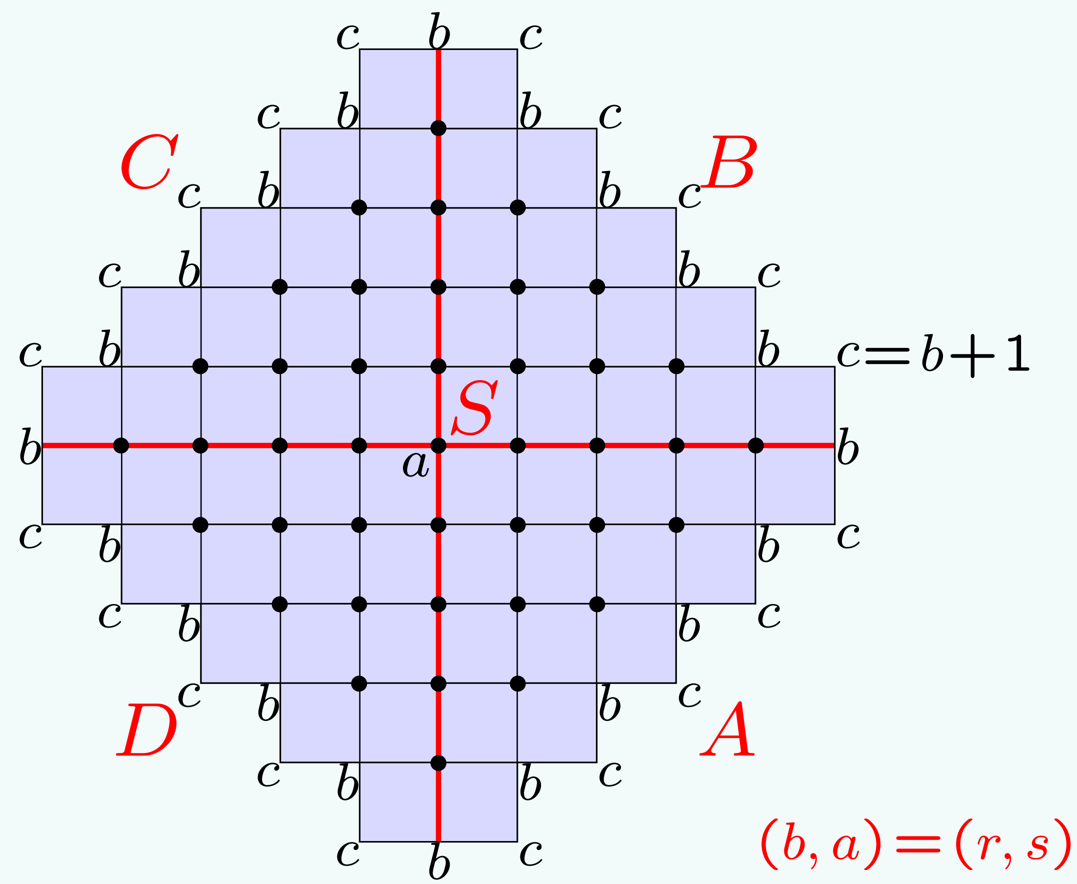
$$P_{r,s} = \langle \delta(a,s) \rangle_r = \lim_{N\to\infty} \frac{\mathrm{Tr}\, SABCD}{\mathrm{Tr} ABCD}$$

where S fixes the center height a to the height s and r labels the boundary conditions with heights $b = r$ and $c = r+1$.

- For the Ising model, the magnetization is

$$m = P_{1,1} - P_{1,3} = \langle \delta(a,1) \rangle_1 - \langle \delta(a,3) \rangle_1$$

with spins $+, 0$ or $0, +$ on the boundary.



- In the continuum scaling limit, the critical RSOS models realize the minimal models $\mathcal{M}(m, m')$ (Belavin-Polyakov-Zamolodchikov 1984) — the simplest rational CFTs. Ising is $\mathcal{M}(3, 4)$.

- The free energy, and thus α is known, for the RSOS models.

- For $m = m' - 1$, the field theories are unitary (ABF models). All face weights are positive.

Minimal Model Operator Content

- In the continuum scaling limit, the critical RSOS models realize the minimal models — the simplest rational CFTs with a finite number of scaling operators. The conformal data is

$$c = 1 - \frac{6(m-m')^2}{mm'}$$

$$\Delta^{m,m'}_{r,s} = \frac{(rm'-sm)^2-(m-m')^2}{4mm'}$$

$$\mathrm{ch}^{m,m'}_{r,s}(q) = \frac{q^{-\frac{c}{24}+\Delta^{m,m'}_{r,s}}}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=-\infty}^{\infty}\Big[q^{k(kmm'+rm'-sm)} - q^{(km+r)(km'+s)}\Big]$$

- Finite Kac tables of conformal weights $\Delta^{m,m'}_{r,s}$ for $\mathcal{M}(3,4)$, $\mathcal{M}(2,5)$, $\mathcal{M}(4,7)$, $\mathcal{M}(5,7)$. The pink boxes (will be) associated with order parameters.

s \ r	1	2
3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$

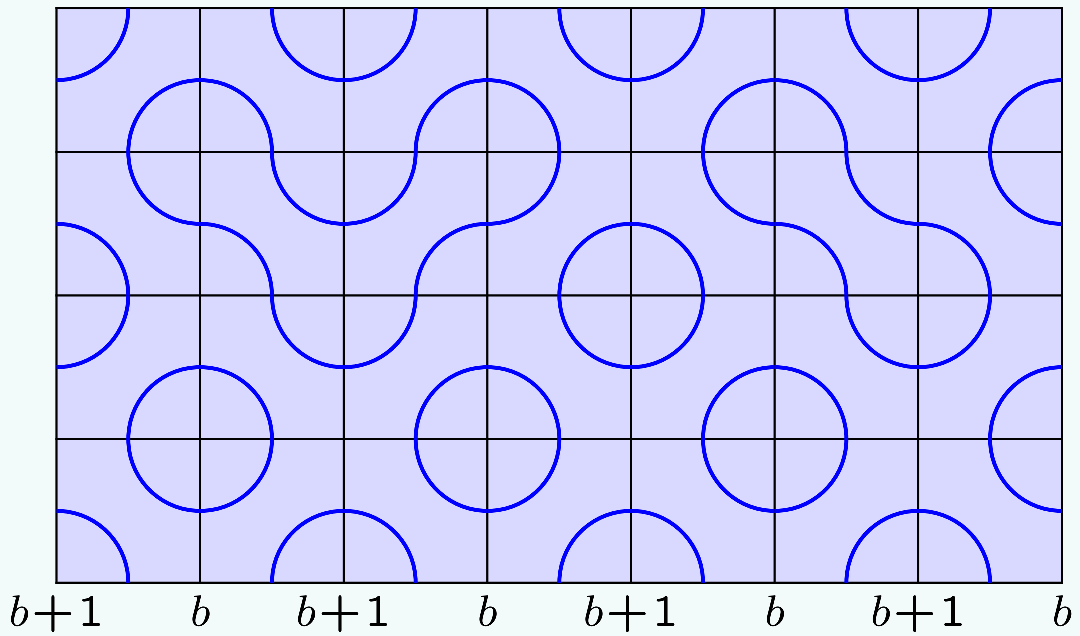
s \ r	1
4	0
3	$-\frac{1}{5}$
2	$-\frac{1}{5}$
1	0

s \ r	1	2	3
6	$\frac{5}{2}$	$\frac{13}{16}$	0
5	$\frac{10}{7}$	$\frac{27}{112}$	$-\frac{1}{14}$
4	$\frac{9}{14}$	$-\frac{5}{112}$	$\frac{1}{7}$
3	$\frac{1}{7}$	$-\frac{5}{112}$	$\frac{9}{14}$
2	$-\frac{1}{14}$	$\frac{27}{112}$	$\frac{10}{7}$
1	0	$\frac{13}{16}$	$\frac{5}{2}$

s \ r	1	2	3	4
6	$\frac{15}{4}$	$\frac{9}{5}$	$\frac{11}{20}$	0
5	$\frac{16}{7}$	$\frac{117}{140}$	$\frac{3}{35}$	$\frac{1}{28}$
4	$\frac{33}{28}$	$\frac{8}{35}$	$-\frac{3}{140}$	$\frac{3}{7}$
3	$\frac{3}{7}$	$-\frac{3}{140}$	$\frac{8}{35}$	$\frac{33}{28}$
2	$\frac{1}{28}$	$\frac{3}{35}$	$\frac{117}{140}$	$\frac{16}{7}$
1	0	$\frac{11}{20}$	$\frac{9}{5}$	$\frac{15}{4}$

Off-Critical Dense Polymers

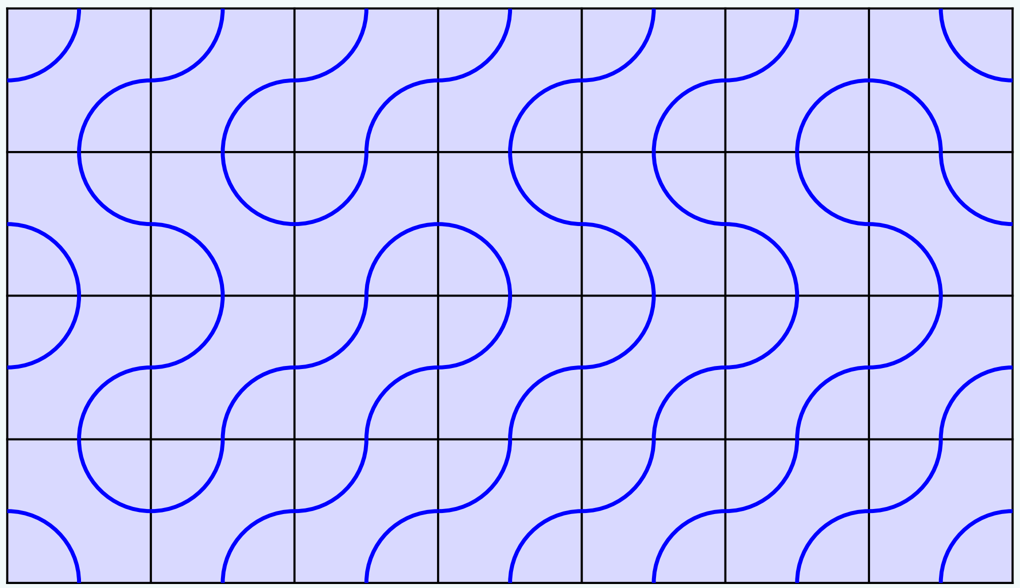
- Typical configurations with heights suppressed:



$T \to 0$ $(t \to 1)$

Small closed loops (no long chains)

Ordered heights (infinite # of flat groundstates)



$T \to T_c$ $(t \to 0)$

Long polymer chains (no closed loops)

Disordered RSOS heights

- The approach to criticality and critical exponents depend on the choice of the groundstate labelled by the Kac label $r = 1, 2, 3, \ldots$ or the height $b = 2, 4, 6, \ldots$.

- This underscores the existence of an infinite number of distinct critical exponents, order parameters and scaling operators — the theory is not rational!

Rasmussen's Logarithmic Limit

- Symbolically, the "*logarithmic limit*" (Rasmussen 2004, 2007) of the minimal CFTs is

$$\lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \mathcal{M}(m,m') = \mathcal{LM}(p,p'), \qquad 1 \le p < p', \quad p,p' \text{ coprime}$$

The limit is taken through coprime pairs (m,m') in the continuum scaling limit, after the thermodynamic limit $N\to\infty$. Extended Kac tables.

- Nontrivial Jordan cells can emerge in this limit, but the equality means identification of the spectra of these CFTs:

$$c^{m,m'} = 1 - \frac{6(m-m')^2}{mm'} \;\to\; 1 - \frac{6(p-p')^2}{pp'} = c^{p,p'}$$

$$\Delta^{m,m'}_{r,s} = \frac{(rm'-sm)^2-(m-m')^2}{4mm'} \;\to\; \frac{(rp'-sp)^2-(p-p')^2}{4pp'} = \Delta^{p,p'}_{r,s}$$

$$\mathrm{ch}^{m,m'}_{r,s}(q) = \frac{q^{-\frac{c}{24}+\Delta^{m,m'}_{r,s}}}{(q)_\infty}\sum_{k=-\infty}^{\infty}\Big[q^{k(kmm'+rm'-sm)} - q^{(km+r)(km'+s)}\Big] \to q^{-\frac{c}{24}+\Delta^{p,p'}_{r,s}}\,\frac{(1-q^{rs})}{(q)_\infty} = \chi^{p,p'}_{r,s}(q)$$

- **New idea:** Apply the logarithmic limit (with care!) to the RSOS models off-criticality

$$\begin{array}{ccc}
\mathcal{M}(m,m') & \xrightarrow{\;t\;} & \mathcal{M}(m,m';t) \\
\text{log}\downarrow & & \text{log}\downarrow \\
\mathcal{LM}(p,p') & \xrightarrow{\;t\;} & \mathcal{LM}(p,p';t)
\end{array}$$

What we did...

- The logarithmic limit of the face weights does not make sense. Instead, we take the logarithmic limit after the thermodynamic limit, in the OPFs.
- One-d configurational sums arise from the diagonalization of the CTMs (ordered limit).
- The conjugate nome q in the 1-d configurational sums is

$$q = e^{-4\pi\lambda/\epsilon} = \text{low-temperature nome}, \qquad t = e^{-\epsilon} = \text{critical nome}$$

Conjugate modulus transformation of elliptic functions $\vartheta_{1,4}(t) \leftrightarrow E(q)$.

- The sums can be interpreted either as Virasoro characters OR as a difference of elliptic functions (ABF approach).
- For $m'-m > 1$, all OPFs diverge at criticality $t \to 0$

$$P_{r,s} \sim (t^{\frac{\pi}{\lambda}})^{\Delta^{m,m'}_{r_0,s_0}}, \qquad \Delta^{m,m'}_{r_0,s_0} = \frac{1-(m'-m)^2}{4mm'} = \underset{(r,s)\in\mathcal{J}}{\text{Min}}\ \Delta^{m,m'}_{r,s} < 0, \qquad m'-m > 1$$

Forrester-Baxter remark that *it is not possible to define order parameters in the usual sense.*

- The logarithmic limit of the order parameters, written in elliptic functions with critical nome, does not make sense.
- BUT the characters also have a conjugate modulus transformation, using the S-matrix.

$$\mathcal{S}_{rs;r's'} = \sqrt{\frac{8}{mm'}}(-1)^{(r'+s')(r+s)} \sin\frac{\pi(m'-m)rr'}{m} \sin\frac{\pi(m'-m)ss'}{m'}$$

One Point Functions and Generalized Order Parameters

$$P_{r,s} = \frac{q^{\frac{c}{24}-\Delta^{m,m'}_{r,s}+\frac{(s-r)(s-r-1)}{4}} E(q^{\frac{s}{2}}, q^{\frac{m'}{2(m'-m)}})(q)_\infty}{E(-q^{\frac{1}{2}},q^2)E(q^{\frac{r}{2}}, q^{\frac{m}{2(m'-m)}})} \, \mathrm{ch}^{m,m'}_{r,s}(q)$$

$$= \sqrt{\frac{2m}{m'}} \frac{\eta(t^{\frac{m'}{m'-m}})\,\vartheta_1(\frac{s\pi(m'-m)}{m'},t)}{\vartheta_4(0,t^{\frac{m'}{m'-m}})\,\vartheta_1(\frac{\pi r(m'-m)}{m},t^{\frac{m'}{m}})} \sum_{(r's')\in\mathcal{J}} \mathcal{S}_{rs;r's'}\,\mathrm{ch}^{m,m'}_{r',s'}(t^{\frac{m'}{m'-m}})$$

- Generalizing Huse 1984, we introduce **Generalized Order Parameters** (GOPs)

$$R_{r'',s''} = \sum_{(r,s)\in\mathcal{J}} \mathcal{S}_{r''s'';rs} \frac{\sin\frac{\pi r(m'-m)}{m}}{\sin\frac{\pi s(m'-m)}{m'}} P_{r,s}$$

Since $\mathcal{S}^2 = I$, the modular matrix $\mathcal{S}$ effectively undoes the modular $\mathcal{S}$ matrix introduced by the conjugate modulus transformation.

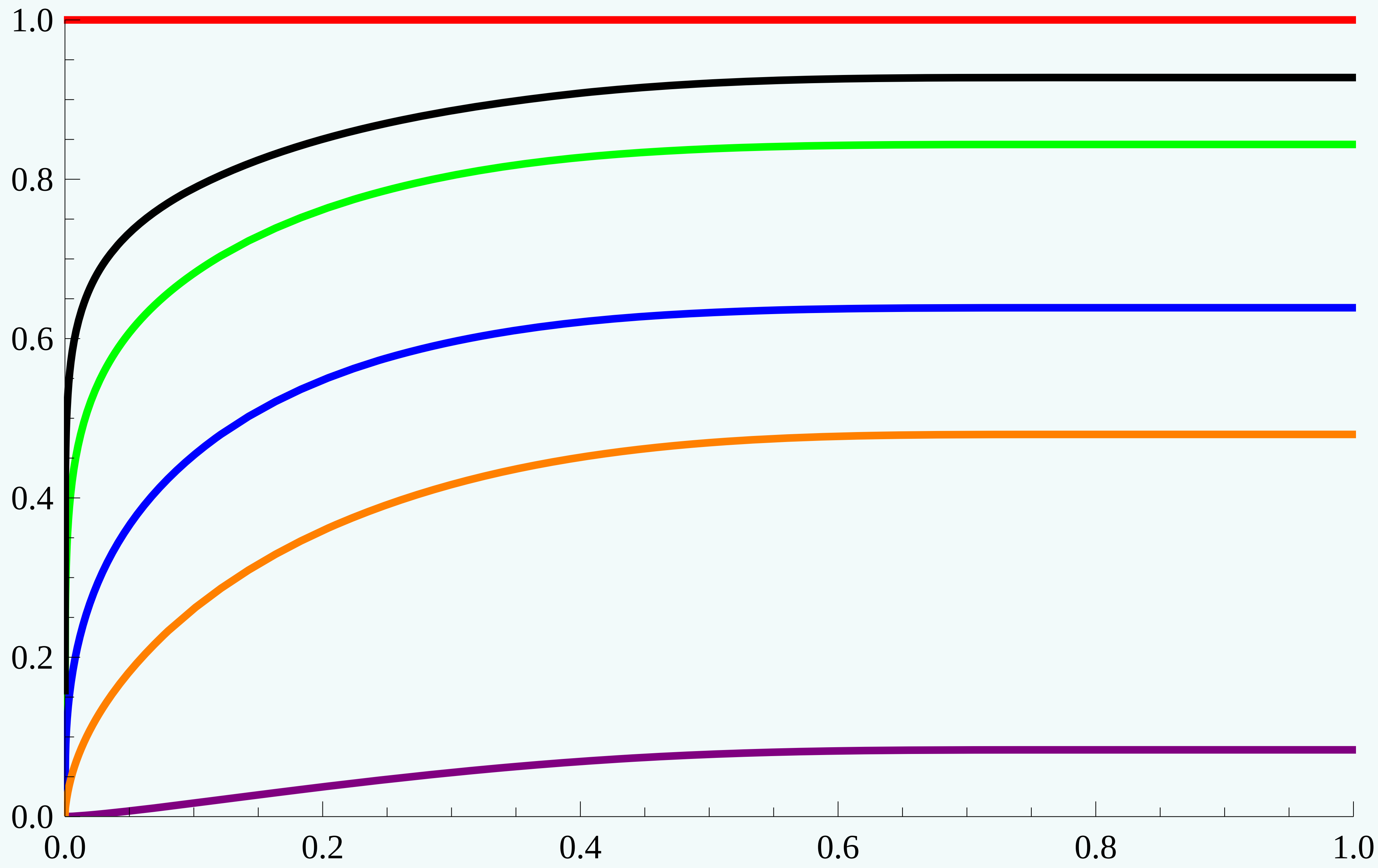
- Defining new observables $\mathcal{O}_{r,s}$ as ratios of the GOPs yields

$$\mathcal{O}_{r,s} = \frac{R_{r,s}}{R_{1,1}} \sim (t^{\frac{\pi}{\lambda}})^{\Delta^{m,m'}_{r,s}} + (t^{\frac{\pi}{\lambda}})^{\Delta^{m,m'}_{r_0,s_0}} O(t^2), \qquad \mathcal{O}_{r,s} \sim (t^2)^{\beta_{r,s}}$$

where t^2 measures the departure from criticality. The second term is of smaller order if (r,s) satisfies $(m'r-ms)^2 < 1+8m(m'-m)$ yielding the critical exponents

$$\beta_{r,s} = (2-\alpha)\Delta^{m,m'}_{r,s} = \frac{(rm'-sm)^2-(m'-m)^2}{8m(m'-m)}, \qquad 2-\alpha = \frac{\pi}{2\lambda} = \frac{m'}{2(m'-m)}$$

Plots of Order Parameters for $\mathcal{M}(4,7)$



- Plot of the observables $\mathcal{O}_{r,s}$, as a function of t, for the minimal model $\mathcal{M}(4,7)$. From top to bottom, we plot $\mathcal{O}_{1,1}, \frac{1}{\mathcal{O}_{2,3}}, \frac{1}{\mathcal{O}_{1,2}}, \mathcal{O}_{1,3}, \mathcal{O}_{2,2}, \mathcal{O}_{1,4}$ corresponding to $|\Delta_{r,s}| = 0, \frac{5}{112}, \frac{1}{14}, \frac{1}{7}, \frac{27}{112}, \frac{9}{14}$ in increasing order with critical exponents $\beta_{r,s} = \frac{7}{6}\Delta_{r,s}$. As expected for order parameters, these observables are nonnegative, vanish at criticality and are increasing functions of t.

Logarithmic Limit of GOPs

- There is no simple conjugate modulus transformation on the infinity of quasi-rational Kac characters $\chi^{p,p'}_{r,s}(q)$. The $\mathcal{S}$ matrix entries have a common prefactor $\sqrt{\frac{8}{mm'}}$ which vanishes in the logarithmic limit so the conjugate modulus transformation is not well behaved.

- Even so, the logarithmic limit of the observables $\mathcal{O}_{r,s}$ (in which the problematic prefactors cancel out in the ratio) are well defined and admit a power series expansion about $t=0$

$$\mathcal{O}^{\infty}_{r,s} = \lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \frac{R_{r,s}}{R_{1,1}} \sim (t^{\frac{\pi}{\lambda}})^{\Delta^{p,p'}_{r,s}} + (t^{\frac{\pi}{\lambda}})^{\Delta^{p,p'}_{r_0,s_0}} O(t^2), \qquad \lambda = \frac{(p'-p)\pi}{p'}$$

- We conclude that, corresponding to the $\varphi_{1,3}$ perturbation off-criticality,

$$\mathcal{O}^{\infty}_{r,s} \sim (t^2)^{\beta_{r,s}}, \qquad \beta_{r,s} = (2-\alpha)\Delta^{p,p'}_{r,s} = \frac{(rp'-sp)^2-(p'-p)^2}{8p(p'-p)}, \qquad 2-\alpha = \frac{\pi}{2\lambda} = \frac{p'}{2(p'-p)}$$

- We have thus constructed limiting observables for all conformal weights satisfying

$$\boxed{\Delta^{p,p'}_{r,s} < \Delta^{p,p'}_{r_0,s_0} + \frac{2(p'-p)}{p'} = \frac{(p'-p)(9p-p')}{4pp'}}$$

These occur for (r,s) in the infinitely extended Kac tables satisfying

$$\boxed{(p'r-ps)^2 < 8p(p'-p)}$$

Summary and Outlook

- In two dimensions, simple statistical systems with local degrees of freedom, such as the Ising model of a magnet, are *rational theories* with a *finite* number of *scaling operators*. The simplest such theories are the *minimal models* $\mathcal{M}(m, m')$ associated with the RSOS lattice models. The operator content and associated critical exponents are encoded in a *finite Kac table* of conformal dimensions.

- Two dimensional systems with nonlocal degrees of freedom, such as *polymers and percolation*, are not rational theories — they are *logarithmic theories* with an infinite number of scaling operators. The simplest such theories are the *logarithmic minimal models* $\mathcal{LM}(p, p')$. An infinite number of scaling operators and associated critical exponents are encoded in an *infinitely extended Kac table*.

- The logarithmic minimal models can be obtained as a limit of the minimal models

$$\lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \mathcal{M}(m, m') = \mathcal{LM}(p, p'), \qquad 1 \le p < p', \quad p, p' \text{ coprime}$$

The logarithmic limit of certain GOPs $\mathcal{O}_{r,s}$ yield critical exponents $\beta_{r,s}$ associated with conformal weights $\Delta_{r,s}^{p,p'}$ of the logarithmic minimal models $\mathcal{LM}(p, p')$ in the infinitely extended Kac table.

- We conclude that *generalized models of polymers and percolation* are exactly solvable both *at criticality and off-criticality!*

- We described the integrable $\varphi_{1,3}$ off-critical perturbation but the $\varphi_{2,1}$ and $\varphi_{1,2}$ perturbations are also integrable by studying *dilute lattice models* (Warnaar et al 1992/94).