A New Look at Nonlinear Dynamical Systems

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Agenda

- **Motivation**
- **Methodology: Canonical dual transformation**
- **Conclusions**

Population Dynamics Modeling

- **Understanding of the dynamics of the species is obtained through fitting models to data collected from biology to estimate vital parameters in population dynamics.**
- **Investigate the population dynamics by the optimization method**
- **Surplus production model is widely used in fisheries and biology**
- Logistic differential equation

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}) - C$$

By finite difference method

$$x_{t+1} = x_t + r x_t (1 - \frac{x_t}{K}) - C_t + E_t$$

- x_t – Stock abundance in year t
- C – Constant harvesting rate
- K – "ideal" population or carrying capacity
- r – Rate of population growth
- E_t – Process error

Optimization Modelling

- The stock biomass x can not be observed, so introduce observation model

$$\hat{I}_t = qx_t = I_t + \varepsilon_t$$

- $\hat{I}_t$ – estimation of abundance
- I_t – abundance observed
- q – proportional coefficient
- ε_t – observation error

- **By least squares method**

$$(P_0) \qquad \min \sum_{t=1}^{n} (qx_t - I_t)^2 + \frac{\alpha}{2} \sum_{t=1}^{n-1} E_t^2$$

$$\text{s.t.} \quad x_{t+1} = x_t + rx_t(1 - \frac{x_t}{K}) - C_t + E_t, t = 1, \cdots, n-1$$

α : penalty factor

Reformulation

Let $I=\{I_t\}\in R^n, C=\{C_t\}\in R^{n-1}, A=\{\delta_{ij}^t\}\in R^{n\times(n-1)\times n}$, and

$$\delta_{ij}^t=\begin{cases}1, \text{if } i=j=t\\ 0, \text{otherwise}\end{cases}$$

$$D=\begin{bmatrix}0 & 1 & \cdots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}, \quad R=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \ddots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 1 & 0\end{bmatrix},$$

$$(P_1)\quad \min\ P_1(x,q,K,r)=\| qx-\frac{I}{K}\|^2+\frac{\alpha}{2}\| rx^T Ax+(D-(1+r)R)x+\frac{C}{K}\|^2$$

$$\text{s.t.}\quad 0<x<1, q>0, K>0, r>0,$$

$$x\in R^n, q\in R, K\in R, r\in R$$

- **Fourth-order polynomial optimization problem with unknown parameters is non-convex**

Dual Transformation

- Stage 1: Fix the parameters q, r, K

$$W(x) = \frac{\alpha}{2} \| rx^T Ax + (D-(1+r)R)x + \frac{C}{K} \|^2 \text{ is nonconvex}$$

- Introduce a geometrical measure

$$\xi = \Lambda_1(x) = rx^T Ax + (D-(1+r)R)x + \frac{C}{K}$$

- Canonical function $V_1(\xi) = \frac{1}{2}\alpha \| \xi \|^2$

- Dual relation $\varsigma = \nabla V_1(\xi) = \alpha\xi$

- Legendre conjugate $V_1^*(\varsigma) = \max\{\xi^T \varsigma - V_1(\xi) \mid \xi \in R^{n-1}\} = \frac{1}{2\alpha}\varsigma^T \varsigma$

- Rewrite inequalty constraint in the canonical form: $x \circ (x-e) < 0$

- Introduce geometrical operator

$$\varepsilon = \Lambda_2(x) = x \circ (x-e)$$

- Indicator function $V_2(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon < 0 \\ +\infty & \text{otherwise} \end{cases}$

- Fenchel transformation

$$V_2^{\#}(\sigma) = \sup\{\varepsilon^T \sigma - V_2(\varepsilon) \mid \varepsilon \in R^n\} = \begin{cases} 0 & \text{if } \sigma \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

- $$P_1(x) = \| qx - \frac{I}{K} \|^2 + V_1(\Lambda_1(x)) + V_2(\Lambda_2(x))$$

- Total complementary function

$$\Xi(x,\varsigma,\sigma) = \| qx - \frac{I}{K} \|^2 + \varsigma^T \Lambda_1(x) - V_1^*(\varsigma) + \sigma^T \Lambda_2(x) - V_2^\#(\sigma)$$

$$= \frac{1}{2} x^T G(\varsigma,\sigma) x - F^T(\varsigma,\sigma) x - V_1^*(\varsigma) - V_2^\#(\sigma)$$

$$G(\varsigma,\sigma) = 2(rA^T\varsigma + q^2 H + \mathrm{Diag}(\sigma)),$$

$$F(\varsigma,\sigma) = 2\frac{q}{K} I + \sigma - (D - (1+r)R)^T \varsigma$$

$$\nabla_x \Xi(x,\varsigma,\sigma) = 0 \Rightarrow$$

$$x = G^{-1}(\varsigma,\sigma) F(\varsigma,\sigma)$$

- **Dual problem**

$$(P^d) \quad \max \; P^d(\varsigma,\sigma) = -\frac{1}{2} F(\varsigma,\sigma) G^{-1}(\varsigma) F(\varsigma,\sigma) - \frac{1}{2\alpha} \varsigma^T \varsigma + \frac{1}{K} C^T \varsigma + \frac{1}{K^2} I^2 I$$

$$\text{s.t.} \quad \varsigma \in R^{n-1}, \sigma \in R^n, \sigma > 0$$

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$$\text{s.t.} \quad \varsigma \in R^{n-1}, \sigma \in R^n, \sigma > 0$$

Thm: Given q, K, r, if (ς^*, σ^*) is a KKT point of (P^{d}), then the vector

$$x^* = G^{-1}(\varsigma^*, \sigma^*) F(\varsigma^*, \sigma^*)$$

is a KKT point of (P_1), and $P(x^*) = P^d(\varsigma^*, \sigma^*)$.

$$S_a^+ = \{(\varsigma, \sigma) \in S_a \mid G(\varsigma, \sigma) \succ 0\}$$

Thm: Given q, K, r, if $(\varsigma^*, \sigma^*) \in S_a^+$ is a critical point of (P^{d}), and $x^* = G^{-1}(\varsigma^*, \sigma^*) F(\varsigma^*, \sigma^*)$. Then x^* is a global minimizer of $P_1(x)$ if and only if (ς^*, σ^*) is a global maximizer of $P^d(\varsigma, \sigma)$, i.e.,

$$P_1(x^*) = \min_{x \in R^n} P_1(x) \Leftrightarrow \max_{(\varsigma, \sigma) \in S_a^+} P^d(\varsigma, \sigma) = P^d(\varsigma^*, \sigma^*)$$

Stage 2

When the global optimal solution of problem (P_1) is obtained by dual method, the optimal parameter q^*, K^*, r^* can be obtained by minimization problem.

$$(P_2) \quad \min P_2(q,K,r) = \| qx^* - \frac{I}{K} \|^2 + \frac{\alpha}{2} \| rx^* A x^* + (D-(1+r)R)x^* + \frac{C}{K} \|^2$$

$$\text{s.t. } q > 0, K > 0, r > 0$$

- By combing together problem (P_1) and (P_2), the global optimal solution (x^*, q^*, r^*, K^*) can be obtained by certain alternative iteration method.

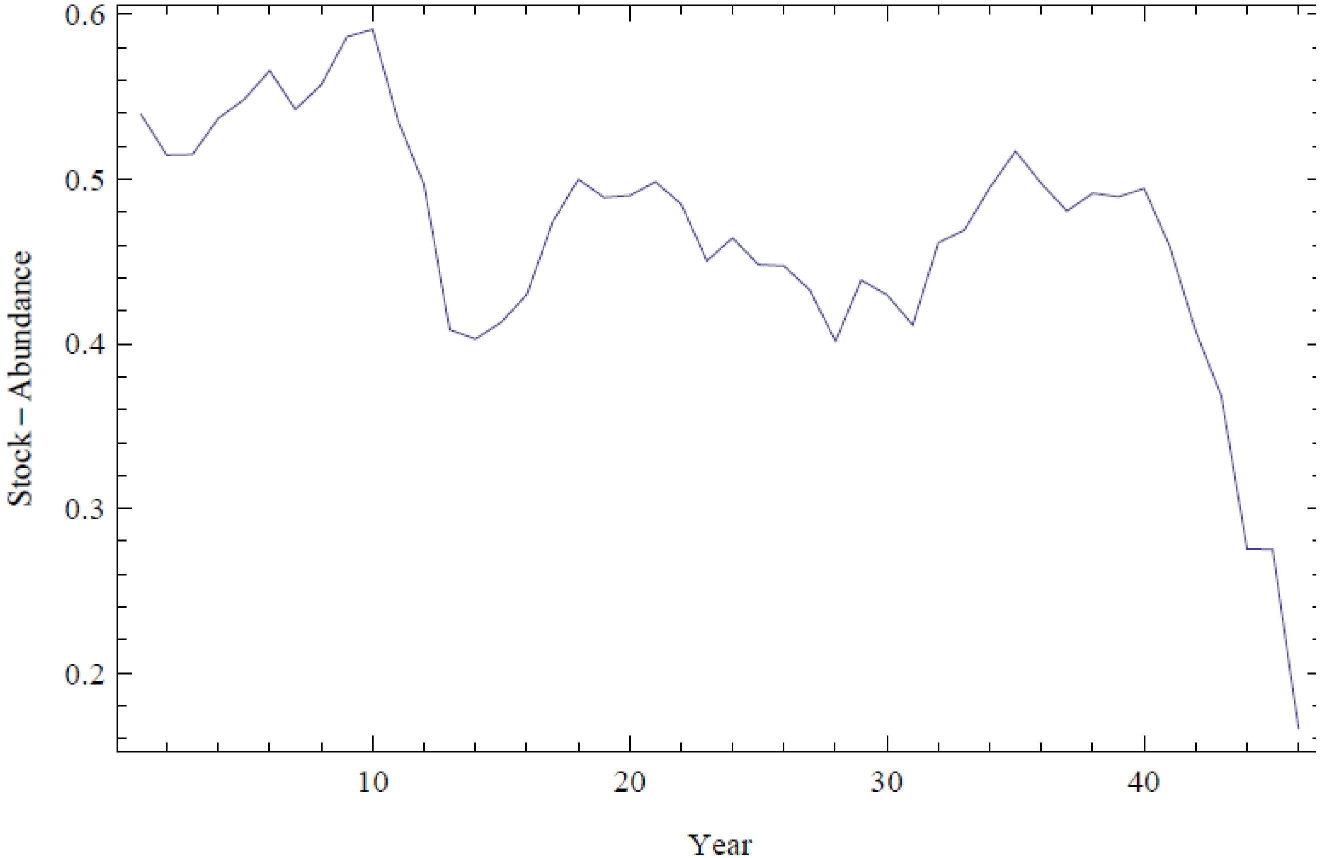
Numerical Experiment

- The data we used is about New Zealand rock lobster over 46 years

- Table :
  - • Catch – Yield C
  - • CUPE – Catch - per - unit - effort I

Year	Catch	CPUE	Year	Catch	CPUE
1945	809	3.49	1968	4975	1.53
1946	854	3.38	1969	4786	1.32
1947	919	3.18	1970	4699	1.45
1948	1360	3.56	1971	4478	1.40
1949	1872	1.79	1972	3495	1.09
1950	2672	4.35	1973	3784	1.23
1951	2834	2.33	1974	3643	1.12
1952	3324	2.57	1975	2987	0.92
1953	4160	2.88	1976	3311	1.02
1954	5541	3.85	1977	3237	1.0
1955	5909	4.16	1978	3418	1.05
1956	6547	4.34	1979	4050	1.09
1957	5049	3.70	1980	4190	1.02
1958	4447	2.37	1981	4058	1.01
1959	4018	2.46	1982	4331	0.98
1960	3762	2.06	1983	4385	1.01
1961	4042	2.21	1984	4911	0.85
1962	4583	2.19	1985	4856	0.84
1963	4554	2.44	1986	4657	0.81
1964	4597	2.14	1987	4500	0.84
1965	4984	2.18	1988	3128	0.68
1966	5295	2.13	1989	3318	0.62
1967	4782	1.86	1990	2770	0.54

- **Stock abundance over 46 years**



Conclusions

■ By combining the methods of the finite difference and the Least Squares, the non-linear problem is reformulated as a non-convex optimization problem with inequality constraints.

■ Due to the non-convexity of the target function with unknown parameters, the problem is difficult to be solve by direct methods.

■ By the canonical duality theory, a global optimal solution is obtained for the first time to the problem over the whole time domain.