Modular Properties of Fractional Level WZW Models

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- DR: arXiv:0810.3532, arXiv:1001.3960, arXiv:1012.2905 [hep-th].
- T Creutzig and DR: arXiv:1205.6513 [hep-th], sequel in preparation...

Ancient History (pre-2000)

Fractional level WZW modules

A few teething problems...

Modern History (post-2000)

Avoiding the mistakes of the past!

The case when $k = -\frac{1}{2}$

Breaking News (circa 2012)

To the future...

The Coset Construction

The coset construction of Goddard, Kent and Olive concretely realises the minimal models $\mathcal{M}(p, p+1)$ $(p = 2, 3, 4, \ldots)$ in terms of unitary Wess-Zumino-Witten models:

$$\mathcal{M}(k+2, k+3) = \frac{\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{sl}}(2)_1}{\widehat{\mathfrak{sl}}(2)_{k+1}}.$$

This proved that these minimal models were unitary and completed the classification of unitary highest weight representations for the Virasoro algebra.

Basically, $\mathcal{M}(k+2, k+3)$ is the commutant of $\widehat{\mathfrak{sl}}(2)_{k+1}$ in $\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{sl}}(2)_1$, hence the minimal model characters may be realised as branching functions.

Fractional Genesis

The complete set of (Virasoro) minimal models is, however, parametrised by two integers:

$$\mathcal{M}\left(p,q\right), \quad p,q = 2,3,4,\ldots \text{ with } p<q \text{ and } \gcd\left\{p,q\right\}=1.$$

These are non-unitary for $q-p>1$.

Comparing central charges suggests (Kent)

$$\mathcal{M}\left(p,q\right) \overset{?}{=} \frac{\widehat{\mathfrak{sl}}\left(2\right)_k \oplus \widehat{\mathfrak{sl}}\left(2\right)_1}{\widehat{\mathfrak{sl}}\left(2\right)_{k+1}}, \qquad k = \frac{3p-2q}{q-p}.$$

Question: Are there fractional level Wess-Zumino-Witten models corresponding to these values of k?

Evidence for Fractional Level WZW Models

If these exist, they are not realised as strings propagating on $\mathsf{SL}(2)$ or $\mathsf{SO}(3)$. Strings on $\mathsf{SL}(2;\mathbb{R})$ or AdS_3 are possibilities...

One consistency requirement for these models is that the partition function should be invariant under modular transformations.

Kac and Wakimoto found a class of admissible (irreducible, highest-weight) representations whose characters carried a representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$.

This class is non-empty if and only if k is of the form required to get a minimal model as a coset.

Adamović and Milas later proved that these admissible representations were precisely the objects in the category $\mathcal{O}$ for the vertex algebra associated with $\widehat{\mathfrak{sl}}(2)_k$.

Beware of overconfidence...

This led to many attempts to "construct" fractional level models from these admissible representations:

- Koh & Sorba (1988),
- Bernard & Felder (1990),
- Mathieu & Walton (1990),
- Awata & Yamada (1992),
- Ramgoolam arXiv:hep-th/9301121,
- Feigin & Malikov arXiv:hep-th/9310004,
- Andreev arXiv:hep-th/9504082,
- Petersen, Rasmussen & Yu arXiv:hep-th/9607129,
- Furlan, Ganchev & Petkova arXiv:hep-th/9608018.

But, while modular invariant partition functions could be found, other consistency requirements were found to be problematic.

Trouble in Paradise

In particular, the relationship between the modular S-matrix $\mathbb{S}$ and fusion did not quite meet expectations:

- The Verlinde formula gave negative (integer) fusion coefficients.
- Computing fusion rules via singular vector decoupling gave different fusion coefficients (with their own problems).
- The conjugation matrix $\mathbb{S}^2$ also contained negative (integer) entries.
- In general, the conjugate of an admissible representation was not itself admissible.

Many *ad hoc* "solutions" proclaimed — but none were universally agreed upon. Di Francesco, Mathieu & Sénéchal wrote that the fractional level theories may possess an "intrinsic sickness".

A New Approach

The category of admissible highest weight representations is not closed under conjugation. A larger category must be sought.

Gaberdiel investigated the closure under fusion for $\widehat{\mathfrak{sl}}(2)_{-4/3}$:

- The fusion of two admissibles can result in an irreducible non-highest weight representation whose conformal dimensions are not bounded below.
- The fusion of an admissible and its conjugate can result in new irreducible non-highest weight representations whose conformal dimensions are bounded below.
- The fusion of these new representations can result in indecomposable modules of logarithmic type.

Adamović & Milas knew about some of these new irreducibles; Feigin, Semikhatov & Tipunin and Maldacena & Ooguri knew about the rest. The indecomposables were new.

An Older Approach

Shortly thereafter, Lesage, Mathieu, Rasmussen & Saleur studied $\widehat{\mathfrak{sl}}(2)_{-1/2}$ using a free field realisation in terms of a lorentzian boson and a pair of symplectic fermions.

They proposed distinct "theories" according as to which, if any, of the fermion fields have antiderivatives. Only two proposals have non-degenerate two-point functions.

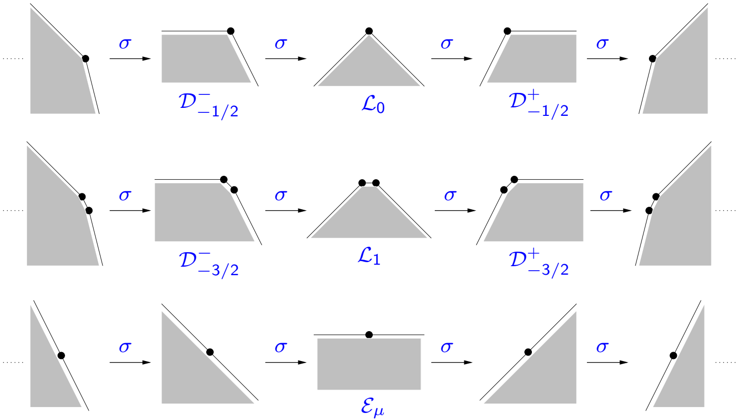
The results suggest logarithmic indecomposables are present in one theory (inherited from symplectic fermions), but not in the other.

Recent work casts doubt on the consistency of the non-logarithmic theory. Therefore, $\widehat{\mathfrak{sl}}(2)_{-1/2}$ is logarithmic, like $\widehat{\mathfrak{sl}}(2)_{-4/3}$.

Example: The Spectrum

For $\widehat{\mathfrak{sl}}(2)_{-1/2}$, the irreducible spectrum is (Adamović–Milas + spectral flow σ):

$$\sigma^\ell(\mathcal{L}_0), \quad \sigma^\ell(\mathcal{L}_1), \quad \sigma^\ell(\mathcal{E}_\mu) \qquad (\ell \in \mathbb{Z},\ \mu \in \mathbb{R}/2\mathbb{Z}).$$



Example: Fusion Rules

The fusion rules are given by (DR)

$$\mathcal{L}_\lambda \times \mathcal{L}_{\lambda'} = \mathcal{L}_{\lambda+\lambda'}, \quad \mathcal{L}_\lambda \times \mathcal{E}_{\mu'} = \mathcal{E}_{\lambda+\mu'}, \quad \mathcal{L}_\lambda \times \mathcal{S}_{\lambda'} = \mathcal{S}_{\lambda+\lambda'},$$

$$\mathcal{E}_\mu \times \mathcal{E}_{\mu'} = \begin{cases} \mathcal{S}_{\mu+\mu'} & \text{if } \mu+\mu' \in \mathbb{Z}, \\ \sigma\left(\mathcal{E}_{\mu+\mu'+1/2}\right) \oplus \sigma^{-1}\left(\mathcal{E}_{\mu+\mu'-1/2}\right) & \text{otherwise.} \end{cases}$$

$$\mathcal{E}_\mu \times \mathcal{S}_{\lambda'} = \sigma^{-2}\left(\mathcal{E}_{\lambda'+\mu+1}\right) \oplus 2\,\mathcal{E}_{\lambda'+\mu} \oplus \sigma^{2}\left(\mathcal{E}_{\lambda'+\mu+1}\right),$$

$$\mathcal{S}_\lambda \times \mathcal{S}_{\lambda'} = \sigma^{-2}\left(\mathcal{S}_{\lambda+\lambda'+1}\right) \oplus 2\,\mathcal{S}_{\lambda+\lambda'} \oplus \sigma^{2}\left(\mathcal{S}_{\lambda+\lambda'+1}\right),$$

where $\lambda, \lambda' \in \mathbb{Z}/2\mathbb{Z}$ and $\mu, \mu' \in \mathbb{R}/2\mathbb{Z}$.

This is extended to the entire spectrum using the well-known conjecture that "fusion respects spectral flow":

$$\sigma^{\ell_1}\left(\mathcal{M}\right) \times \sigma^{\ell_2}\left(\mathcal{N}\right) = \sigma^{\ell_1+\ell_2}\left(\mathcal{M} \times \mathcal{N}\right).$$

Example: Logarithmic Indecomposables

The $\mathcal{S}_\lambda$ ($\lambda = 0, 1$) generated by fusion are logarithmic indecomposables uniquely determined by their structure diagrams.

$$
\begin{tikzcd}
& \mathcal{L}_0 \arrow[dl] \arrow[dr] & \\
\sigma^{-2}(\mathcal{L}_1) \arrow[dr] & \mathcal{S}_0 & \sigma^{2}(\mathcal{L}_1) \arrow[dl] \\
& \mathcal{L}_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{L}_1 \arrow[dl] \arrow[dr] & \\
\sigma^{-2}(\mathcal{L}_0) \arrow[dr] & \mathcal{S}_1 & \sigma^{2}(\mathcal{L}_0) \arrow[dl] \\
& \mathcal{L}_1 &
\end{tikzcd}
$$

The representation is logarithmic because the Virasoro mode L_0 acts non-semisimply with rank 2 Jordan blocks. This leads to correlators with logarithmic singularities.

Example: Characters

The characters of the admissible irreducibles are

$$\mathrm{ch}\left[\mathcal{L}_0\right] = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z;q)} + \frac{\eta(q)}{\vartheta_3(z;q)}\right], \qquad \mathrm{ch}\left[\mathcal{E}_\mu\right] = \frac{z^\mu}{\eta(q)^2}\sum_{n\in\mathbb{Z}} z^{2n},$$

$$\mathrm{ch}\left[\mathcal{L}_1\right] = \frac{1}{2}\left[\frac{\eta(q)}{\vartheta_4(z;q)} - \frac{\eta(q)}{\vartheta_3(z;q)}\right],$$

supplemented by

$$\mathrm{ch}\left[\sigma^\ell(\mathcal{M})\right](z;q) = z^{\ell k} q^{\ell^2 k/4}\mathrm{ch}\left[\mathcal{M}\right](zq^{\ell/2};q).$$

One therefore gets periodicities:

$$\mathrm{ch}\left[\sigma^{\ell-1}(\mathcal{L}_\lambda)\right] + \mathrm{ch}\left[\sigma^{\ell+1}(\mathcal{L}_{\lambda+1})\right] = 0.$$

Among the $\sigma^\ell(\mathcal{L}_\lambda)$, there are only four independent characters!

A Quotient Ring

The modules

$$\sigma^{\ell-1}(\mathcal{L}_\lambda) \oplus \sigma^{\ell+1}(\mathcal{L}_{\lambda+1}), \quad \sigma^{\ell}(\mathcal{E}_\mu), \quad \sigma^{\ell}(\mathcal{S}_\lambda)$$

span an ideal of the fusion ring. The quotient *Gr* is free of rank 4 and is isomorphic to the character ring of the $\sigma^{\ell}(\mathcal{L}_\lambda)$.

A basis is given by the admissible highest weight representations:

$$[\mathcal{L}_0], \quad [\mathcal{L}_1], \quad [\mathcal{D}^+_{-1/2}], \quad [\mathcal{D}^+_{-3/2}].$$

The fusion product descends to *Gr* as:

	$[\mathcal{L}_0]$	$[\mathcal{L}_1]$	$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-3/2}]$
$[\mathcal{L}_0]$	$[\mathcal{L}_0]$	$[\mathcal{L}_1]$	$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-3/2}]$
$[\mathcal{L}_1]$	$[\mathcal{L}_1]$	$[\mathcal{L}_0]$	$[\mathcal{D}^+_{-3/2}]$	$[\mathcal{D}^+_{-1/2}]$
$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-1/2}]$	$[\mathcal{D}^+_{-3/2}]$	$-[\mathcal{L}_1]$	$-[\mathcal{L}_0]$
$[\mathcal{D}^+_{-3/2}]$	$[\mathcal{D}^+_{-3/2}]$	$[\mathcal{D}^+_{-1/2}]$	$-[\mathcal{L}_0]$	$-[\mathcal{L}_1]$

Modular Transformations

With $z = \mathrm{e}^{2\pi\mathrm{i}\zeta}$, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\mathbb{S}\colon\ (\zeta|\tau) \mapsto (\zeta/\tau|-1/\tau)$, we find

$$\mathbb{S} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & \mathfrak{i} & \mathfrak{i} \\ -1 & -1 & \mathfrak{i} & \mathfrak{i} \end{pmatrix}, \quad \mathbb{S}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The negative coefficients in the conjugation matrix are natural because, *eg*. the conjugate of $\mathcal{D}^{+}_{-1/2}$ is $\mathcal{D}^{-}_{1/2} = \sigma^{-1}(\mathcal{L}_0)$ and

$$\mathrm{ch}\left[\sigma^{-1}(\mathcal{L}_0)\right] = -\mathrm{ch}\left[\sigma(\mathcal{L}_1)\right] = -\mathrm{ch}\left[\mathcal{D}^{+}_{-3/2}\right].$$

Similarly, the negative coefficients appearing in the product on *Gr* are precisely those obtained from the Verlinde formula.

Where are we now?

We have seen that the failure of the ancients to reconcile fusion via singular vector decoupling with the Verlinde formula has two main sources:

- An unsatisfactory choice of representation category.
- The map from the fusion ring to the character ring has a very large kernel.

We seem to have fixed the category problem, introducing uncountably many additional irreducibles. What can we do about the second?

The key is the observation (Feigin, Semikhatov & Tipunin; Lesage, Mathieu, Rasmussen & Saleur) that the characters of the $\sigma^{\ell}(\mathcal{L}_{\lambda})$ do not converge everywhere in the z-plane.

The $\mathrm{ch}\big[\sigma^{\ell}(\mathcal{L}_{\lambda})\big]$ have poles which limit their convergence regions to annuli in z.

The periodicity $\mathrm{ch}\big[\sigma^{\ell-1}(\mathcal{L}_{\lambda})\big] + \mathrm{ch}\big[\sigma^{\ell+1}(\mathcal{L}_{\lambda+1})\big] = 0$ relates characters with disjoint convergence regions, hence must be understood in the sense of meromorphic continuation.

As formal power series, such sums are not zero. In fact (DR),

$$\mathrm{ch}\big[\sigma^{\ell-1}(\mathcal{L}_{\lambda})\big] + \mathrm{ch}\big[\sigma^{\ell+1}(\mathcal{L}_{\lambda+1})\big] = \mathrm{ch}\big[\sigma^{\ell}(\mathcal{E}_{\lambda+1/2})\big].$$

Moreover, the latter may be interpreted as a distribution supported at the poles of the $\mathrm{ch}\big[\sigma^{\ell}(\mathcal{L}_{\lambda})\big]$.

In this distributional setting, the (Grothendieck) fusion ring is isomorphic to the character ring. Modular transformations are known and the Verlinde formula works for $k = -\frac{1}{2}$ and $k = -\frac{4}{3}$ (Creutzig & DR). This is being generalised to all admissible k.

Future Directions

- This new paradigm for fractional level WZW models is being checked (Canagasabey & DR) against the motivating coset story.
- Homological aspects of the new category need to be settled, *eg.* the $\mathcal{S}_\lambda$ and the irreducible $\mathcal{E}_\mu$ are expected to be projective and injective.
- Our success suggests that a detailed algebraic understanding of the $\mathsf{SL}(2;\mathbb{R})$ (AdS_3) WZW model may be in reach (*cf.* Maldacena & Ooguri).
- This is one instance of our programme to rewrite the role of modular transformations in logarithmic conformal field theory. The examples currently occupying the field are pathological in that they should instead be understood through better-behaved subtheories and the magic of simple currents.

Merci!

et bon appetit...