Boundary Conditions for Infinitely Extended Kac Table of Critical Dense Polymers

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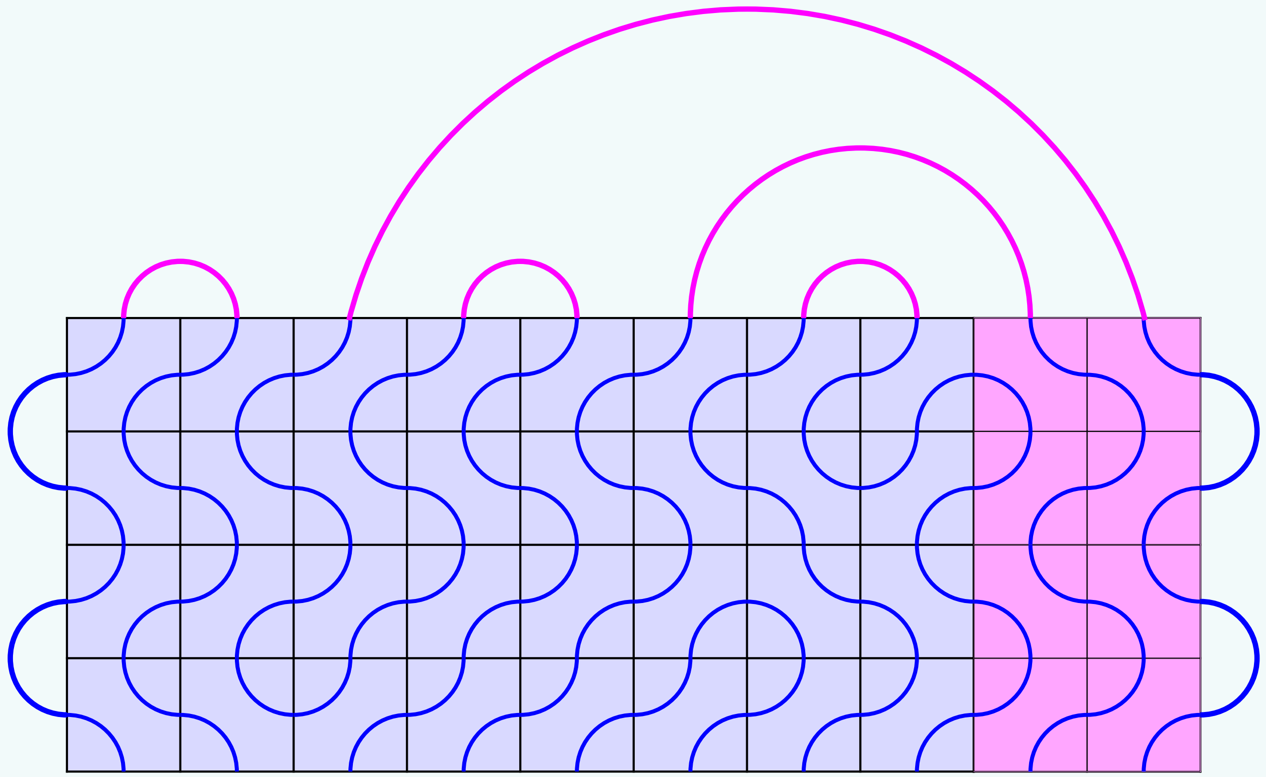

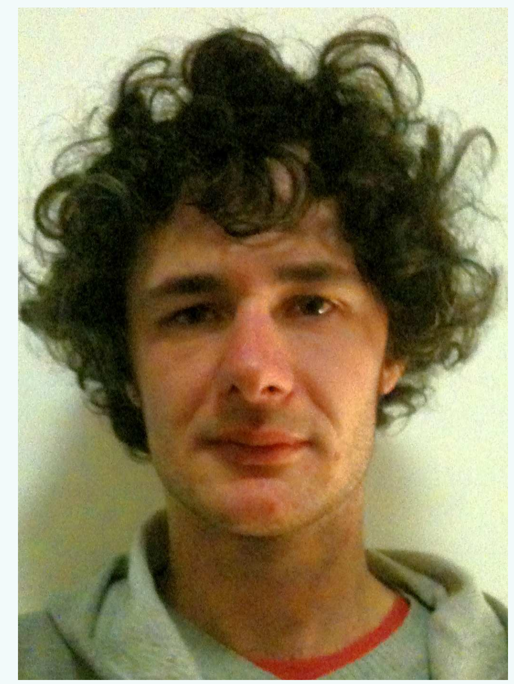

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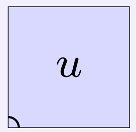
Outline

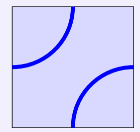
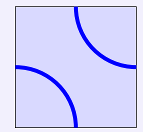
Critical dense poymers is exactly solvable
on *finite* strips for lattice implementations
of all (r, s) conformal boundary conditions!

- Lattice model $\mathcal{LM}(1,2)$
- Double-row transfer matrices and Yang-Baxter integrability
- Boundary Yang-Baxter Equation (BYBE)
- Link states
- r-type boundary operators
- Inversion identities
- Finite-size corrections
- Selection rules
- q-Catalan polynomials
- Conformal data and infinitely extended Kac table

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- **Logarithmic Minimal Models:** Yang-Baxter integrable loop models on the square lattice. Face operators defined in planar Temperley-Lieb algebra (Jones 1999)

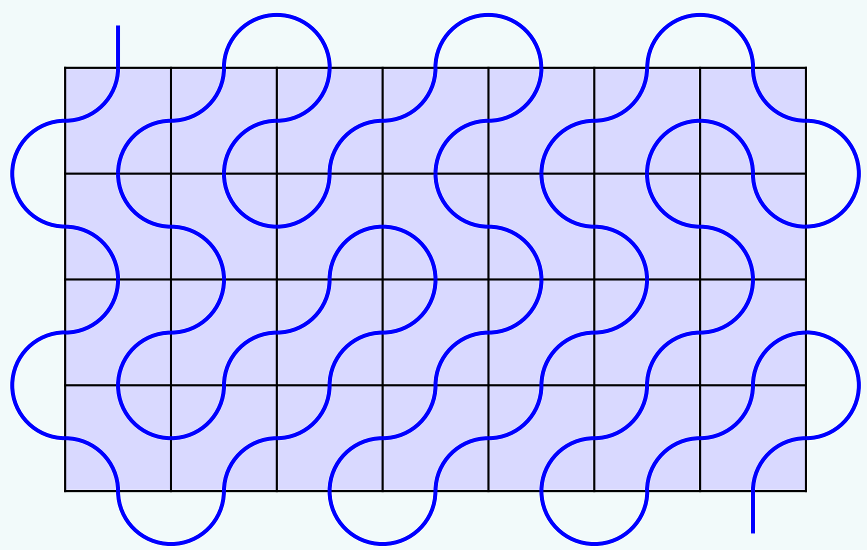
$X(u) =$ 

$= \sin(\lambda - u)$  $+ \sin u$  $= \sin(\lambda - u)\, I + \sin u\, e_j$

$1 \le p < p'$ coprime integers, $\qquad \lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

$u =$ spectral parameter, $\qquad \beta = 2\cos\lambda =$ nonlocal loop fugacity

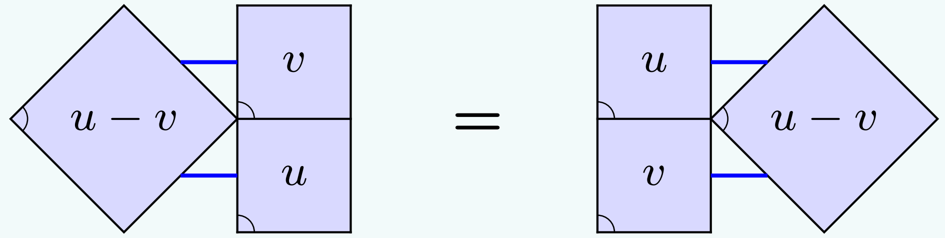
- **Critical Dense Polymers $\mathcal{LM}(1,2)$:** $\qquad (p,p') = (1,2), \qquad \lambda = \dfrac{\pi}{2}$



$\boxed{\beta = 0 \;\Rightarrow\; \text{no loops} \;\Rightarrow\; \text{space filling dense polymer}}$

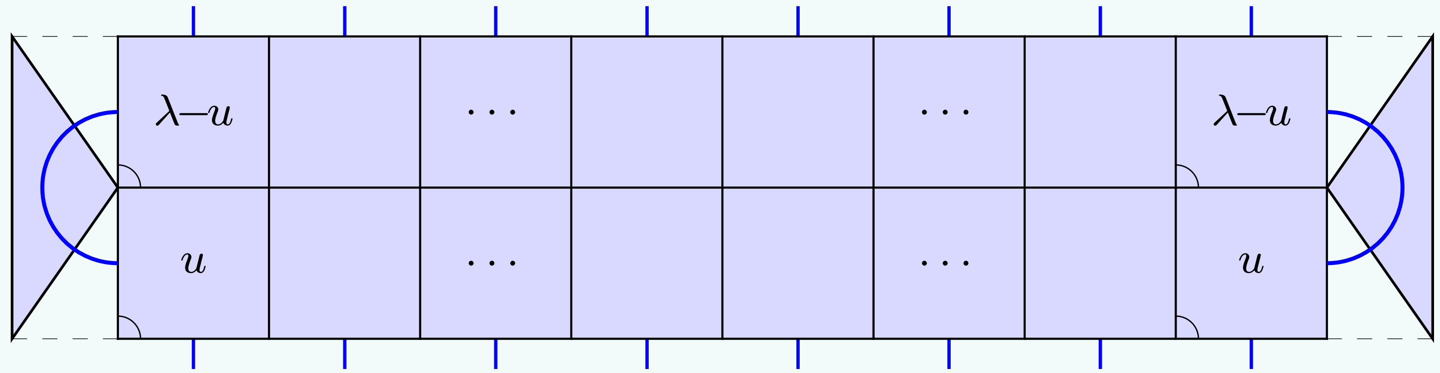
Yang-Baxter Integrability

Yang-Baxter Equation



Double-row transfer matrices in $(r,s)=(1,1)$ sector

$$\boldsymbol{D}(u) \;=\; \frac{1}{\sin 2u}$$

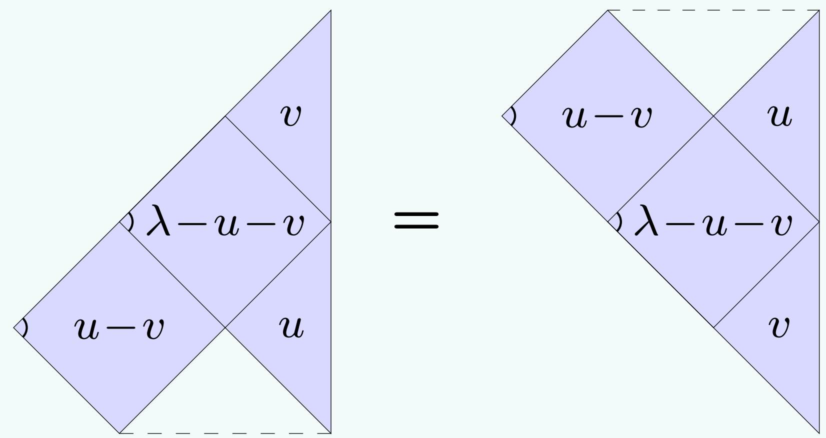


$$\boxed{\text{YBE} + \text{BYBE} \quad \Rightarrow \quad \boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u)}$$

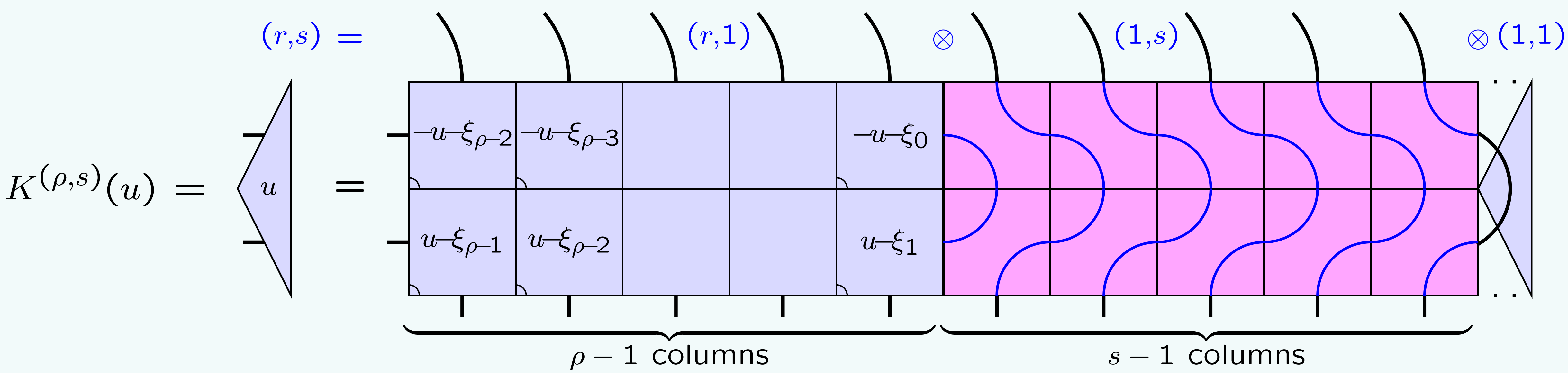
- Multiplication is vertical concatenation of diagrams.
- Equality is the equality of N-tangles.
- Act on link states to obtain a matrix representation.

Boundary Yang-Baxter Equation

- The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary 2-tangles



- For $r, s = 1, 2, 3, \ldots$, the (r, s) BYBE solution is built as the fusion product of $(r, 1)$ and $(1, s)$ integrable seams acting on the vacuum $(1, 1)$ triangle:



- The s-type seam introduces $s-1$ defects.
- The r-type seams are realized with $r = \lceil \frac{\rho}{2} \rceil$, that is, either $\rho = 2r-1$ or $\rho = 2r$.
- The column inhomogeneities are: $\xi_k = (k + \frac{1}{2})\lambda$

Link States

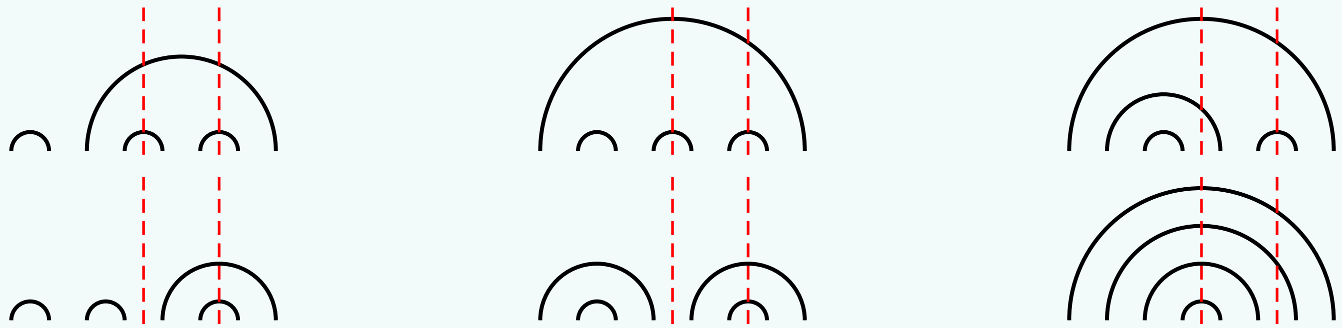
- For (r,s) boundary conditions, $\boldsymbol{D}(u)$ acts on a vector space of link states $\mathcal{V}^{(N)}_{\rho,s}$ restricted so that there are no half-arcs closing in the $\rho-1$ columns of the r-type seam and no half-arcs closing in the $s-1$ columns of the s-type seam. Half-arcs are allowed to close *between* the r- and s-type seams.

- The dimension of $\mathcal{V}^{(N)}_{\rho,s}$ with $N+\rho+s-2$ nodes is

$$\dim \mathcal{V}^{(N)}_{\rho,s} = \binom{N}{\frac{N-\rho+s}{2}} - \binom{N}{\frac{N-\rho-s}{2}} = \begin{cases} \binom{N}{\frac{N-2r+s}{2}} - \binom{N}{\frac{N-2r-s}{2}}, & \rho = 2r \\ \binom{N}{\frac{N-2r+s+1}{2}} - \binom{N}{\frac{N-2r-s+1}{2}}, & \rho = 2r-1 \end{cases}$$

These are generalized Catalan numbers.

- The six link states for $\mathcal{V}^{(N)}_{3,3}$ with $N=4$ are



r-Type Boundary Operators

- The face operators are $\quad X_j(u) = s(\lambda-u)I + s(u)e_j, \qquad s(u) = \dfrac{\sin u}{\sin\lambda}, \qquad \beta = 2\cos\lambda$
- The r-type boundary operators $K_0^{(\rho)}(u,\xi)$ are solutions to the BYBE constructed as

$$\boxed{\begin{aligned} K_0^{(\rho)}(u,\xi) &= \frac{1}{\eta^{(\rho)}(u,\xi)} \prod_{j=0}^{\rho-2} X_j(u-(\rho-j-1)\lambda-\xi) \prod_{j=\rho-2}^{0} X_j(u+(\rho-j-1)\lambda+\xi) \\ &= I + \frac{s(2u)}{s(u+\xi)s\big(u-\rho\lambda-\xi\big)} \sum_{k=0}^{\rho-2} (-1)^k s\big((\rho-k-1)\lambda\big) e_0^{(k)} \end{aligned}}$$

where $\eta^{(\rho)}(u,\xi)$ is a normalization. The proof of this identity is by induction.

- $K_0^{(\rho)}(u,\xi)$ is restricted to act from the space of link states with no closed half-arcs in the $\rho-1$ right-most columns onto itself. A basis in this algebra is

$$\text{basis} = \{I, e_0^{(k)}\}, \qquad e_0^{(k)} = e_0 e_1 \cdots e_{k-1} e_k = \prod_{j=0}^{k} e_j = \text{ordered products}$$

- If $U_\rho(\frac{\beta}{2})$ are Chebyshev polynomials of order ρ, the generalized TL projectors are

$$P_j^{(\rho)} = \sum_{k=0}^{\rho-2} (-1)^k s\big((\rho-k-1)\lambda\big) e_j^{(k)} = \sum_{k=0}^{\rho-2} (-1)^k U_{\rho-k-2}(\tfrac{\beta}{2})\, e_j^{(k)}, \qquad P_j^{(\rho)}\, P_j^{(\rho')} = U_{\rho-1}(\tfrac{\beta}{2})\, P_j^{(\rho')}$$

$$\boxed{P_j^{(2)} = e_j, \qquad P_j^{(3)} = \beta e_j - e_j e_{j+1}, \qquad P_j^{(4)} = (\beta^2-1)e_j - \beta e_j e_{j+1} + e_j e_{j+1} e_{j+2}}$$

Inversion Identities

- In all (r,s) sectors, the eigenvalues $d(u)$ of the normalized double-row transfer matrices

$$\boldsymbol{d}(u)=\begin{cases}\dfrac{2^{\rho-1}\boldsymbol{D}(u)}{\sin 2u\cos^{\rho-2}2u}, & \rho=2r\\[2ex] \dfrac{2^{\rho-1}\boldsymbol{D}(u)}{\sin 2u\cos^{\rho-1}2u}, & \rho=2r-1\end{cases}$$

satisfy the *universal* inversion identities (Baxter 1982, OPW 1996, PR 2007)

$$d(u)d(u+\lambda)\;=\begin{cases}\Big(\cos^{2N}u+\sin^{2N}u\Big)^2, & \rho=2r\\[2ex] \left(\dfrac{\cos^{2N}u-\sin^{2N}u}{\cos^2 u-\sin^2 u}\right)^2, & \rho=2r-1\end{cases}$$

subject to the initial condition and crossing symmetry

$$d(0)=1,\qquad d(\lambda-u)=d(u)$$

- The inversion identities are proved directly in the planar algebra. The solutions take the form

$$d(u)=\begin{cases}\displaystyle\prod_{j=1}^{\lfloor\frac{N-1}{2}\rfloor}\Big(1+\epsilon_j\sin\tfrac{j\pi}{N}\sin 2u\Big)\Big(1+\mu_j\sin\tfrac{j\pi}{N}\sin 2u\Big), & s\text{ odd}\\[3ex] \displaystyle\prod_{j=1}^{\lfloor\frac{N}{2}\rfloor}\Big(1+\epsilon_j\sin\tfrac{(2j-1)\pi}{2N}\sin 2u\Big)\Big(1+\mu_j\sin\tfrac{(2j-1)\pi}{2N}\sin 2u\Big), & s\text{ even}\end{cases}$$

where $\epsilon_j^2=\mu_j^2=1$ for all j. Note that double zeros occur if $\epsilon_j=\mu_j$.

Finite-Size Corrections

- The partition function for a $P \times N$ strip is

$$Z_{P,N} = \mathrm{Tr}\, \boldsymbol{d}(u)^P = \sum_n d_n(u)^P = \sum_n e^{-PE_n(u)}$$

$$E_n(u) = -\ln d_n(u) = 2N f_{bulk}(u) + f_{bdy}(u) + \frac{2\pi \sin 2u}{N}\Big(-\frac{c}{24} + \Delta_{r,s} + k\Big) + \cdots$$

where $f_{bulk}(u)$ and $f_{bdy}(u)$ are the bulk and boundary free energies and $k = 0, 1, 2 \ldots$.

- Euler-Maclaurin gives the central charge $c = -2$ and the (r, s) finite excitations

$$\boxed{-\frac{c}{24} + \Delta_{r,s} + k = \begin{cases} -\frac{c}{24} + \sum\limits_{j \in \mathcal{E}_n} j + \sum\limits_{j \in \mathcal{M}_n} j, & s \text{ odd} \\ -\frac{c}{24} - \frac{1}{8} + \sum\limits_{j \in \mathcal{E}_n} (j - \frac{1}{2}) + \sum\limits_{j \in \mathcal{M}_n} (j - \frac{1}{2}), & s \text{ even} \end{cases}}$$

- The sets

$$\mathcal{E}_n = \{j : \epsilon_j = -1\}, \qquad \mathcal{M}_n = \{j : \mu_j = -1\}$$

encode the location of excited zeros in the analyticity strip in the complex u-plane

$$u_j = \frac{\pi}{4} \pm \frac{i}{2} \ln \tan \frac{\pi E_j}{2N}, \qquad E_j = \begin{cases} j, & s \text{ odd} \\ j - \frac{1}{2} & s \text{ even} \end{cases}$$

These *selection rules* are determined *empirically* based on *physical combinatorics*.

- The lowest excitation $(k = 0)$ in each sector gives

$$\boxed{\Delta_{r,s} = \frac{(2r - s)^2 - 1}{8}, \qquad r, s = 1, 2, 3, \ldots}$$

Selection Rules

- For (r,s) boundary conditions, with N columns and $\rho+s-2$ boundary columns, the combinatorial selection rules single out the following *finitized* partition functions

$$s \text{ odd}: Z^{(N)}_{(1,1)|(r,s)}(q)=\begin{cases} q^{-\frac{c}{24}}\sum_{k=1}^{s} C_{\frac{N-1}{2},\frac{\rho-s-1+2k}{2}}(q), & \rho=2r\\ q^{-\frac{c}{24}}\sum_{k=1}^{s}\Big[C_{\frac{N-2}{2},\frac{\rho-s+2k}{2}}(q)+q^{\frac{N}{2}}C_{\frac{N-2}{2},\frac{\rho-s-2+2k}{2}}(q)\Big], & \rho=2r-1\end{cases}$$

$$s \text{ even}: Z^{(N)}_{(1,1)|(r,s)}(q)=\begin{cases} q^{-\frac{c}{24}-\frac{1}{8}}\sum_{k=1}^{s/2} C'_{\frac{N}{2},\frac{\rho-s-2+4k}{2}}(q), & \rho=2r\\ q^{-\frac{c}{24}-\frac{1}{8}}\sum_{k=1}^{s/2}\Big[C'_{\frac{N-1}{2},\frac{\rho-s-1+4k}{2}}(q)+q^{\frac{N}{2}}C'_{\frac{N-1}{2},\frac{\rho-s-3+4k}{2}}(q)\Big], & \rho=2r-1\end{cases}$$

where $C_{M,r}(q)$ and $C'_{M,r}(q)$ are q-Catalan polynomials and the modular nome is

$$q=e^{-2\pi\tau},\qquad \tau=\frac{P}{N}\sin 2u,\qquad \frac{P}{N}=\text{aspect ratio}$$

- Simplifying, using combinatorial q-identities, gives Kac characters

$$Z^{(N)}_{(1,1)|(r,s)}(q)=\chi^{(N)}_{r,s}(q)=\begin{cases} q^{-\frac{c}{24}+\Delta_{r,s}}\left(\begin{bmatrix} N\\ \frac{N-2r+s}{2}\end{bmatrix}_q-q^{rs}\begin{bmatrix} N\\ \frac{N-2r-s}{2}\end{bmatrix}_q\right), & \rho=2r\\ q^{-\frac{c}{24}+\Delta_{r,s}}\left(\begin{bmatrix} N\\ \frac{N-2r+s+1}{2}\end{bmatrix}_q-q^{rs}\begin{bmatrix} N\\ \frac{N-2r-s+1}{2}\end{bmatrix}_q\right), & \rho=2r-1\end{cases}$$

$$\to \chi_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)},\qquad N\to\infty$$

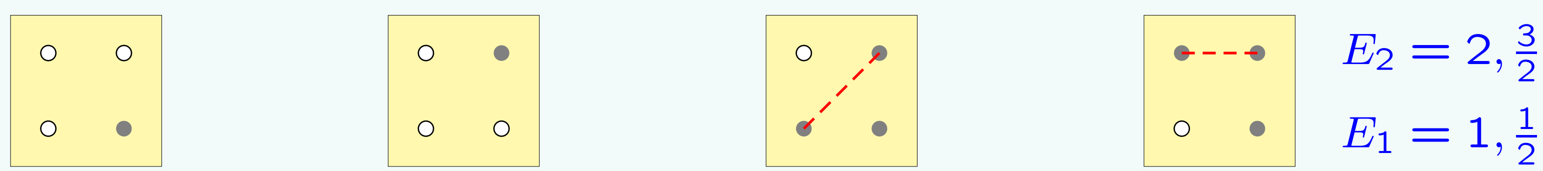
q-Catalan Polynomials

- The q-Catalan polynomials are the spectrum generating functions of the building blocks for the finitized characters in the (r,s) sectors. They are finitized characters for the irreducible representations.

- Explicitly, in terms of q-binomials, the q-Catalan polynomials are

$$\boxed{\begin{aligned} C_{M,r}(q) &= q^{\frac{r(r-1)}{2}} \frac{(1-q^r)}{(1-q^{M+1})} \begin{bmatrix} 2M+2 \\ M+1-r \end{bmatrix}_q \\ C'_{M,r}(q) &= q^{\frac{(r-1)^2}{2}} \frac{(1-q^{2r})}{(1-q^{M+r+1})} \begin{bmatrix} 2M+1 \\ M+1-r \end{bmatrix}_q \end{aligned}}$$

- The q-Catalan polynomials admit a combinatorial interpretation in terms of the patterns of zeros. For $M=2$, $r-1=1$ this combinatorial interpretation is illustrated by



$$\begin{aligned} C_{2,2}(q) &= q + q^2 + q^4 + q^5 \\ C'_{2,2}(q) &= q^{1/2} + q^{3/2} + q^{5/2} + q^{7/2} \end{aligned}$$

The sum $C_{M,r}(q) = \sum_S q^{E(S)}$ is over all admissible double-column configurations S of height M with $r-1$ more occupied sites in the right column.

Conformal Data and Kac Table

- **Central charge:** $(p, p') = (1, 2)$

$$c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; -2$$

- **Infinitely extended Kac table of conformal weights:**

$$\begin{aligned}\Delta_{r,s} \;&=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &=\; \frac{(2r-s)^2-1}{8}, \qquad r,s = 1,2,3,\ldots\end{aligned}$$

- **Kac characters:**

$$\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

$$q \;=\; \exp(-2\pi\tau) \;=\; \text{modular parameter}$$

$$\tau \;=\; \frac{P}{N}\sin 2u \;=\; \text{geometric factor}$$

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Summary

- A Yang-Baxter integrable model of critical dense polymers is *solved exactly* on *finite-width* strips.

- Fixing ρ even or odd, we have constructed an infinite family of integrable boundary conditions labelled by $r, s = 1, 2, 3, \ldots$ which are conjugate to scaling fields in the infinitely extended Virasoro Kac table.

- A study of the *physical combinatorics* of eigenstates, via the eigenvalue patterns of zeros in the complex u-plane, yields finitized characters involving q-Catalan polynomials.

- Solution of the inversion identity allows the finite-size conformal properties to be obtained exactly by Euler-Maclaurin yielding the central charge, conformal weights and characters of a logarithmic CFT with spectrum

$$\boxed{c = -2, \qquad \Delta_{r,s} = \frac{(2r-s)^2 - 1}{8}, \qquad r, s = 1, 2, 3, \ldots}$$

- The conformal dimensions agree with the values obtained from off-critical order parameters (Seaton's talk).