Lax representations of reductions of nonautonomous lattice equations

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in collaboration with
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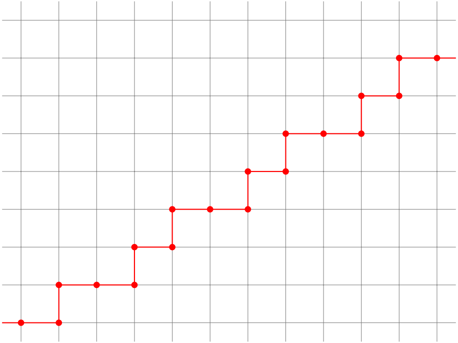
Abstract

We present a method of determining a Lax representation for similarity reductions of autonomous and non-autonomous partial difference equations. We present reductions of one of the simplest integrable lattice equations, the lattice potential Korteweg-de Vries equation, which will give rise to a simple Quispel-Roberts-Thompson map, a discrete version of the first Painlevé equation and a discrete version of the fourth Painlevé equation, and their Lax pairs.

We consider lattice equations of the form

$$Q(w_{l,m}, w_{l+1,m}, w_{l,m+1}, w_{l+1,m+1}; \alpha_l, \beta_m) = 0 \tag{1}$$

where Q is multilinear and multidimensionally consistent and α_l and β_m are functions of l and m respectively. Given a staircase of initial values, one evolves the system by imposing (1) on each square in $\mathbb{Z}^2$.



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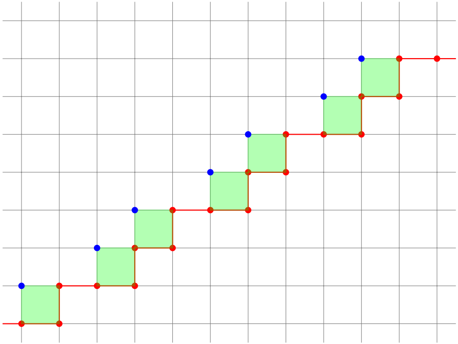

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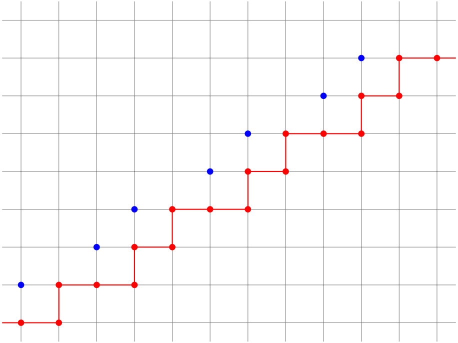

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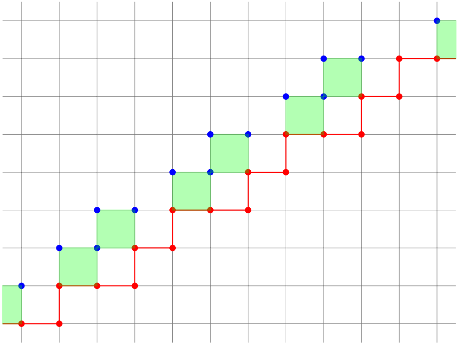

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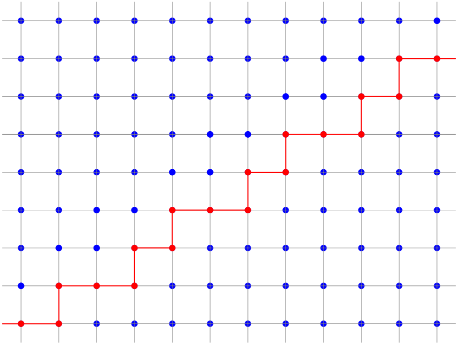

With this initial data, we may find $w_{l,m}$ for all $(l, m) \in \mathbb{Z}^2$.

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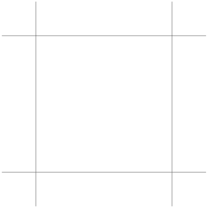

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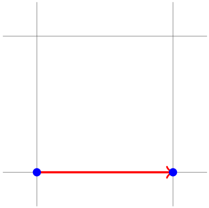

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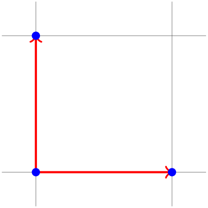

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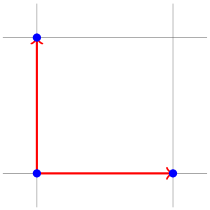

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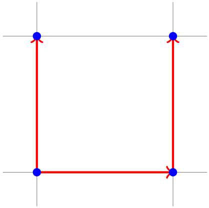

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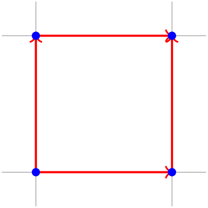

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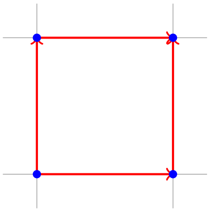

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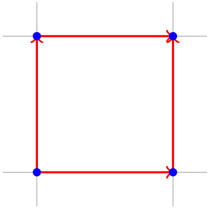

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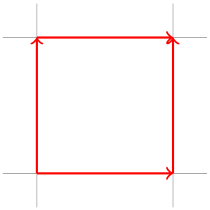

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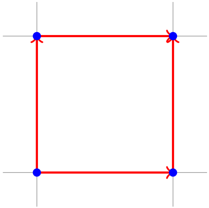

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Implies our given lattice equation

$$Q(w_{l,m}, w_{l+1,m}, w_{l,m+1}, w_{l+1,m+1}; \alpha_l, \beta_m) = 0$$

Our toy example for this talk will be lattice KdV

$$(w_{l,m} - w_{l+1,m+1})(w_{l+1,m} - w_{l,m+1}) = \alpha_l - \beta_m$$

which has the Lax pair

$$L_{l,m} = \left(\begin{array}{cc} w_{l,m} & \alpha_l - \gamma - w_{l,m} w_{l+1,m} \\ 1 & -w_{l+1,m} \end{array} \right), \tag{2}$$

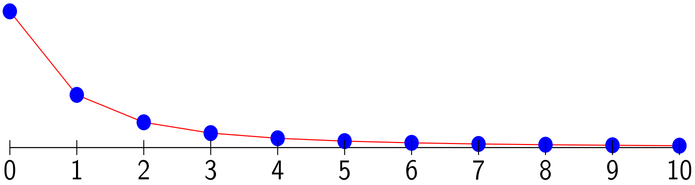
$$M_{l,m} = \left(\begin{array}{cc} w_{l,m} & \beta_m - \gamma - w_{l,m} w_{l,m+1} \\ 1 & -w_{l,m+1} \end{array} \right). \tag{3}$$

Equating $L_{l,m+1} M_{l,m}$ with $M_{l+1,m} L_{l,m}$ gives us the above.

Travelling Wave Reductions

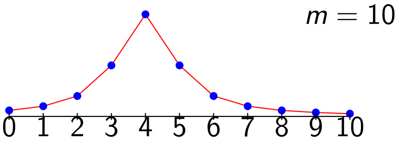
Let us consider a travelling wave

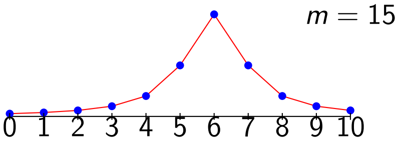
$m = -2$



Travelling Wave Reductions

If we examine two different states, say $m = 10$ and $m = 15$, we see





the solution satisfies a similarity constraint, in this case

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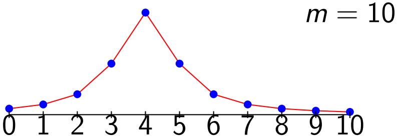

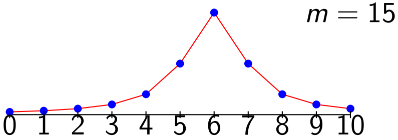

the solution satisfies a similarity constraint, in this case

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so that after some time (in m), the wave has just translated in space. In general, we will consider reductions of the form

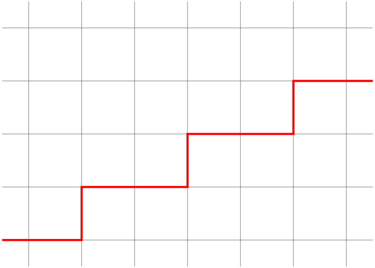
$$w_{l+z_1,m+z_2} = w_{l,m}.$$

which we call a (z_1, z_2)-reduction.

If we take a simplest case of,

$$(w_{l,m} - w_{l+1,m+1})(w_{l+1,m} - w_{l,m+1}) = \alpha_l - \beta_m$$

where $\alpha_l = \alpha = \mathrm{const}$ and $\beta_m = \beta = \mathrm{const}$, then impose a $(2,1)$-reduction, then we obtain a mapping from the



We start by tracing a path going through points sharing the same value.

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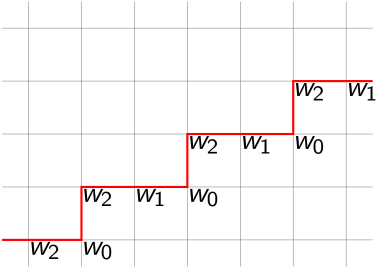

We label the points on the path.

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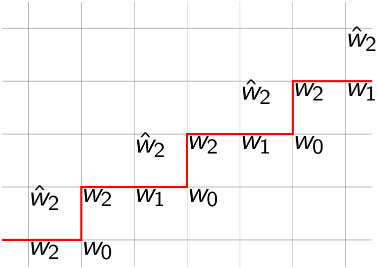

We use the equation on the lattice to define our new point

$$\hat{w}_2 = w_0 + \frac{\alpha - \beta}{w_2 - w_1}.$$

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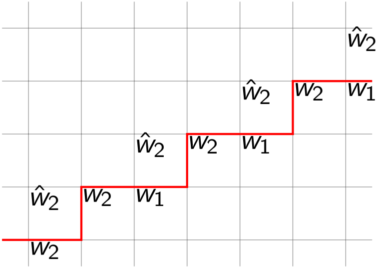

So we may remove values we don't need anymore.

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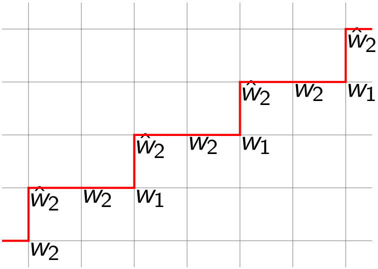

We now have a new set of initial conditions that is the same as our last, but shifted in the $(1,1)$.

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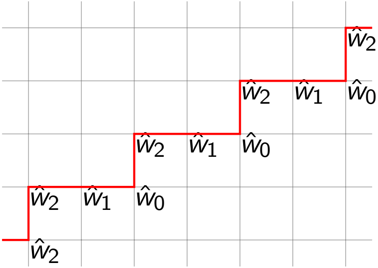

Hence, we have formed a map

$$\phi \begin{pmatrix} w_2 \\ w_1 \\ w_0 \end{pmatrix} = \begin{pmatrix} \hat{w}_2 = w_0 + \dfrac{\alpha - \beta}{w_1 - w_2} \\ \hat{w}_1 = w_2 \\ \hat{w}_0 = w_1 \end{pmatrix}$$

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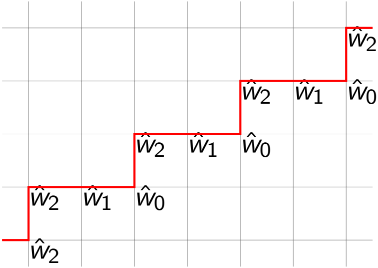

If we allow $n = 2m - l$, then this is in the general form

$$w_{n+3} - w_n = \frac{\alpha - \beta}{w_{n+2} - w_{n+1}}.$$

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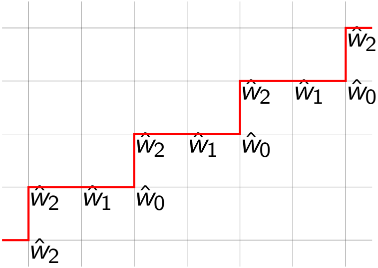
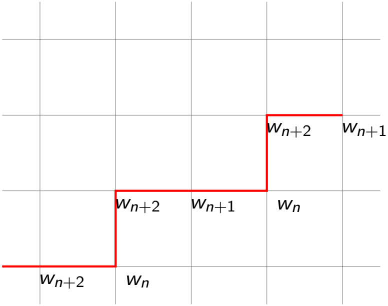


If we let $y_n = w_{n+1} - w_n$, this becomes

$$y_{n+1} + y_n + y_{n-1} = \frac{\alpha - \beta}{y_n}$$

which is a mapping of Quispel-Roberts-Thompson type.

We wish to form a Lax pair for this system



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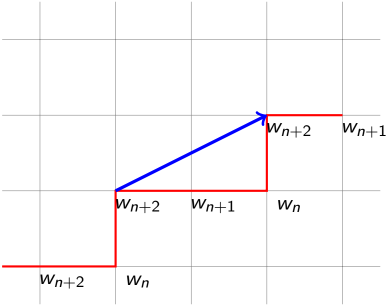

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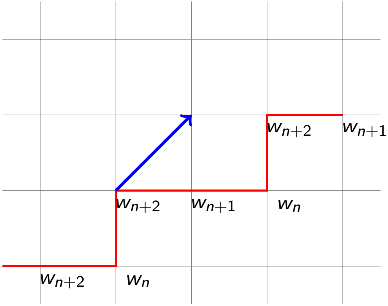

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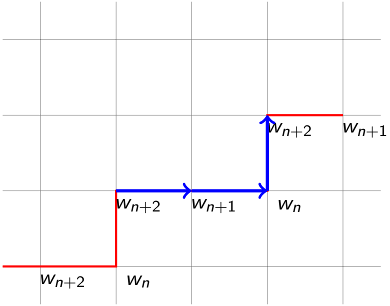

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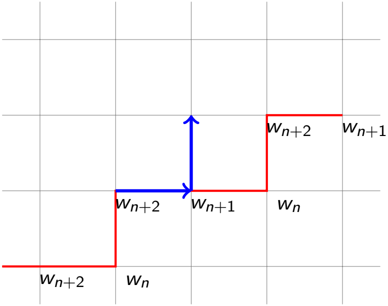

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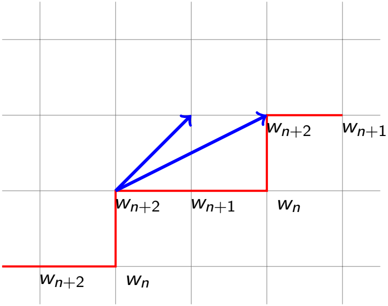

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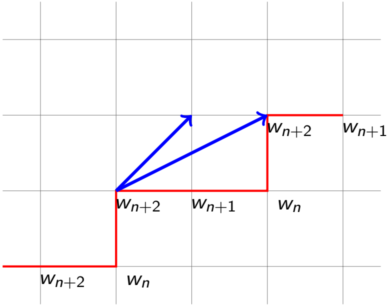

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$$A_n = \begin{pmatrix} w_n & \beta - \gamma - w_n w_{n+2} \\ 1 & -w_{n+2} \end{pmatrix} \begin{pmatrix} w_{n+1} & \alpha - \gamma - w_n w_{n+1} \\ 1 & -w_n \end{pmatrix} \begin{pmatrix} w_{n+2} & \alpha - \gamma - w_{n+1} w_{n+2} \\ 1 & -w_{n+1} \end{pmatrix}$$

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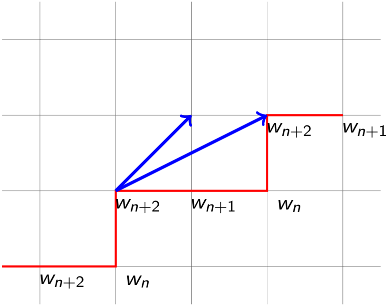

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$$B_n(\gamma) \equiv M_{l+1,m} L_{l,m}$$

Where by $\equiv$ we mean under the variables of the reduction:

$$A_n = \begin{pmatrix} w_n & \beta - \gamma - w_n w_{n+2} \\ 1 & -w_{n+2} \end{pmatrix} \begin{pmatrix} w_{n+1} & \alpha - \gamma - w_n w_{n+1} \\ 1 & -w_n \end{pmatrix} \begin{pmatrix} w_{n+2} & \alpha - \gamma - w_{n+1} w_{n+2} \\ 1 & -w_{n+1} \end{pmatrix}$$

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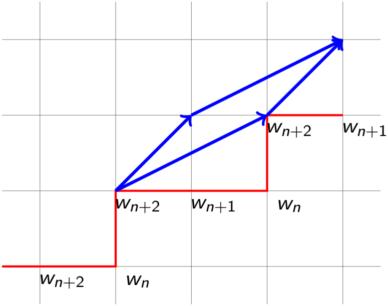

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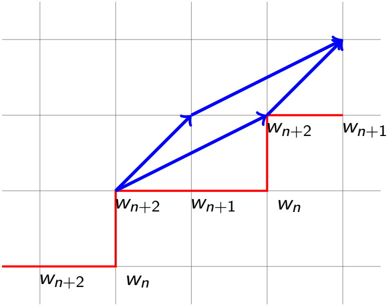

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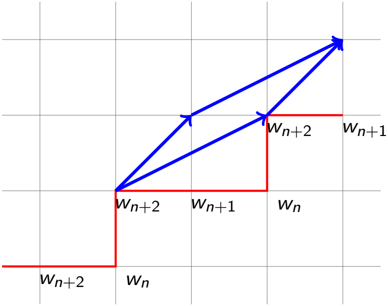

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$$w_{n+3} - w_n = \frac{\alpha - \beta}{w_{n+2} - w_{n+1}}$$

Note: We could express this in terms of the y_n (with some effort).

$$A_n = \begin{pmatrix} w_n & \beta - \gamma - w_n w_{n+2} \\ 1 & -w_{n+2} \end{pmatrix} \begin{pmatrix} w_{n+1} & \alpha - \gamma - w_n w_{n+1} \\ 1 & -w_n \end{pmatrix} \begin{pmatrix} w_{n+2} & \alpha - \gamma - w_{n+1} w_{n+2} \\ 1 & -w_{n+1} \end{pmatrix}$$

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Deautonomizing the theory

If we impose

$$w_{l+z_1,m+z_2} = w_{l,m},$$

we find a constraint that the equation describing the evolution at (l, m) must be the same as the equation at $(l + z_1, m + z_2)$. In our toy example, imposing the above gives

$$(w_{l,m} - w_{l+1,m+1})(w_{l+1,m} - w_{l,m+1}) - \alpha_l + \beta_m = 0,$$
$$(w_{l,m} - w_{l+1,m+1})(w_{l+1,m} - w_{l,m+1}) - \alpha_{l+z_1} + \beta_{m+z_2} = 0.$$

is consistent when

$$\alpha_{l+z_1} - \alpha_l = \beta_{m+z_2} - \beta_m := \mathrm{lcm}(z_1, z_2)h$$

where h is constant in l and m by a seperation of variables argument.

In our example, the $(2, 1)$-reduction, we have

$$\alpha_{l+2} - \alpha_l = \beta_{m+1} - \beta_m = 2h$$

The simplest solution is where these are just linear;

$$\alpha_l = lh + a, \qquad\qquad \beta_m = mh + b.$$

Our reduction is then (remembering $n = 2m - l$)

$$w_{n+3} - w_n = \frac{\alpha_{l+1} - \beta_m}{w_{n+2} - w_{n+1}} = \frac{a - b - hn + h}{w_{n+2} - w_{n+1}}$$

which by letting $y_n = w_{n+1} - w_n$ we obtain the equation

$$y_{n+1} + y_n + y_{n-1} = \frac{nh + b - a}{y_n},$$

which is a discrete version of Painlevé I.

To form the Lax representation here, we find in the original Lax pair, the parameters, α_l and β_m, always appear with γ,

$$L_{l,m} = \begin{pmatrix} w_{l,m} & \alpha_l - \gamma - w_{l,m}w_{l+1,m} \\ 1 & -w_{l+1,m} \end{pmatrix}, \tag{4}$$

$$M_{l,m} = \begin{pmatrix} w_{l,m} & \beta_m - \gamma - w_{l,m}w_{l,m+1} \\ 1 & -w_{l,m+1} \end{pmatrix}. \tag{5}$$

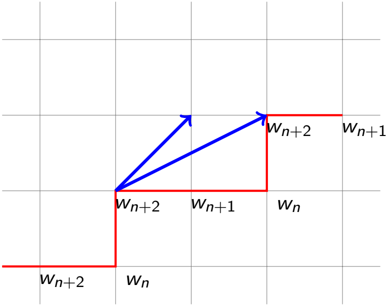
The key part is that we introduce a spectral parameter that couples a direction in (l, m)-space with γ, we let

$$x = lh - \gamma.$$

now the Lax pairs can be written in terms of x and n, as

$$\begin{aligned} \alpha_l - \gamma &= a + \underbrace{hl - \gamma}_{x} \\ \beta_m - \gamma &= b + \underbrace{\beta_m - \alpha_l}_{nh} + \underbrace{hl - \gamma}_{x}. \end{aligned}$$

We wish to form a Lax pair for this system



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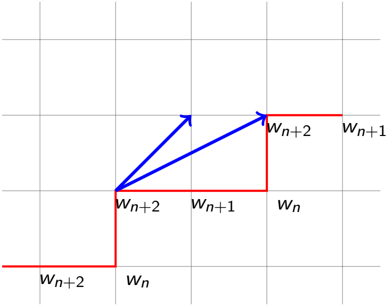

$$Y_n(x+2h) = A_n(x)Y_n(x)$$

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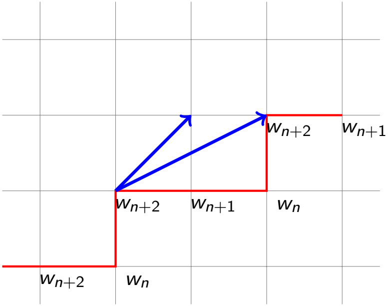

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Where we need to now take into account the non-autonomy:

$$A_n = \begin{pmatrix} w_n & b + nh + x - w_n w_{n+2} \\ 1 & -w_{n+2} \end{pmatrix} \begin{pmatrix} w_{n+1} & x + a + h - w_n w_{n+1} \\ 1 & -w_n \end{pmatrix} \begin{pmatrix} w_{n+2} & a + x - w_{n+1} w_{n+2} \\ 1 & -w_{n+1} \end{pmatrix}$$

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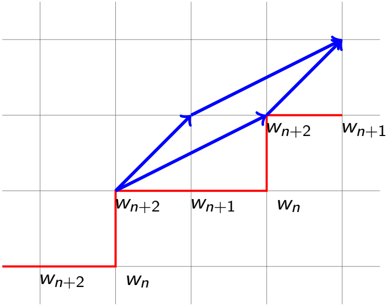

$$Y_{n+1}(x+3h) = A_{n+1}(x+h)B_n(x)Y_n(x$$

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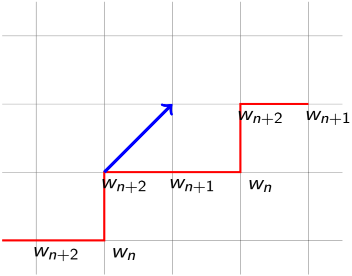
If we examine our original condition,

$$\alpha_{l+2} - \alpha_l = \beta_{m+1} - \beta_m = 2h,$$

a little more closely, we find that we may build an extra variable. The general solution is

$$\alpha_l = \begin{cases} hl + a_1 & \text{where } l \text{ is odd,} \\ hl + a_2 & \text{where } l \text{ is even,} \end{cases}$$

and $\beta_m = 2mh + b$, but since the resulting equation only depends on $\alpha_l - \beta_m$, b can be chosen to be 0.



The single shift, $n \to n+1$ has the effect

$$(w_n - w_{n+3})(w_{n+1} - w_{n+2}) = hn + a_2.$$

$$a_1 \to a_2 + h, \qquad a_2 \to a_1 - h.$$

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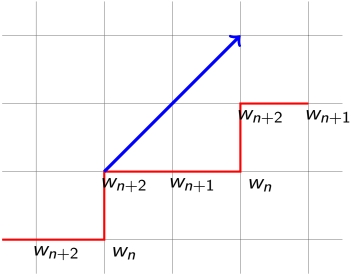

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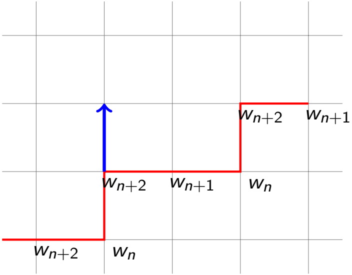

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The double shift is then given by the coupled equations

$$
\begin{aligned}
(w_n - w_{n+3})(w_{n+1} - w_{n+2}) &= hn + a_2, \\
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However, if we let

$$y_n = (w_{n+2} - w_n)(w_{n+2} - w_{n+1}) - a_2,$$
$$z_n = \frac{(w_{n+2} - w_n)(y_n + a_1)}{y_n + nh},$$

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then the system is equivalent to

$$y_n + y_{n+2} = z_n^2 - (a_1 + a_2),$$
$$z_n z_{n+2} = -\frac{(y_{n+2} + a_1)(y_{n+2} + a_2)}{(y_{n+2} + (n+2)h)}.$$

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then the system is equivalent to

$$
(y + \tilde{y} + a_1 + a_2)(y + \underset{\sim}{y} + a_1 + a_2) = \frac{(y + a_1)^2 (y + a_2)^2}{(y + nh)^2}
$$

where $y_n = y$, $y_{n+2} = \tilde{y}$, $y_{n-2} = \underset{\sim}{y}$ and $\tilde{n} = n + 2$. This is a discrete version of the fourth Painlevé equation.

The Lax pair in the y_n and z_n variables are given by

$$\begin{aligned} Y_n(x+2h) &= A_n(x)Y_n(x), \\ Y_{n+2}(x) &= B_n(x)Y_n(x), \end{aligned}$$

where

$$A_n(x) \equiv \begin{pmatrix} -\frac{(y_n+a_1)(y_n+a_2)}{z_n} & x^2+\delta x+\epsilon \\ x-y_n & (y_n+nh)z_n \end{pmatrix},$$

where

$$\begin{aligned} \delta =& y_n+a_1+a_2+hn, \\ \epsilon =& (y_n+hn)(y_n+a_1)+(hn+y_n+a_1)a_2, \end{aligned}$$

and

$$B_n(x) = \begin{pmatrix} -z_n & x+hn+z_n^2 \\ 1 & -z_n \end{pmatrix}.$$

We have also performed a reduction from the lattice modified Korteweg-de Vries equation

$$\alpha_l(w_{l,m}w_{l+1,m} - w_{l,m+1}w_{l+1,m+1}) - \beta_m(w_{l,m}w_{l,m+1} - w_{l+1,m}w_{l+1,m+1}) = 0$$

to a q-analogue of the sixth Painlevé equation

$$y\hat{y} = \frac{q^2\left(q^2 b_1 t^2 + z a_2\right)\left(b_2 t^2 + z a_1\right)}{\left(z b_1 q^2 + a_2\right)\left(a_1 + z b_2\right)},$$
$$z\hat{z} = \frac{\left(t^2 b_1 q^4 + \hat{y} a_1\right)\left(t^2 b_2 q^4 + \hat{y} a_2\right)}{q^2\left(a_1 + \hat{y} b_1\right)\left(a_2 + \hat{y} b_2\right)},$$

where $\hat{t} = q^2 t$ for some fixed $q \in \mathbb{C}$ and the a_i and b_j are fixed parameters.

A reduction from the lattice Schwarzian Korteweg-de Vries equation

$$\alpha_l \left(\frac{1}{w_{l,m+1} - w_{l+1,m+1}} + \frac{1}{w_{l+1,m} - w_{l,m}} \right) = \beta_m \left(\frac{1}{w_{l+1,m+1} - w_{l+1,m}} + \frac{1}{w_{l,m} - w_{l,m+1}} \right), \tag{6}$$

to a q-analogue of the sixth Painlevé equation

$$y\hat{y} = \frac{b_1 b_2 (z-1)\left(q^2 z - 1\right)}{q^2 \left(b_1 b_2 t - \theta_1 z\right)\left(b_1 b_2 t - \theta_2 z\right)}, \tag{7}$$

$$z\hat{z} = \frac{\left(b_1 q^2 t \hat{y} - 1\right)\left(b_2 q^2 t \hat{y} - 1\right)}{q^2 \left(a_1 \hat{y} - 1\right)\left(a_2 \hat{y} - 1\right)}, \tag{8}$$

where $\hat{t} = q^2 t$ for some fixed $q \in \mathbb{C}$ and the a_i, b_j and θ_i are fixed parameters.

Another reduction from the lattice Schwarzian Korteweg-de Vries equation

$$\alpha_l \left(\frac{1}{w_{l,m+1} - w_{l+1,m+1}} + \frac{1}{w_{l+1,m} - w_{l,m}} \right) = \beta_m \left(\frac{1}{w_{l+1,m+1} - w_{l+1,m}} + \frac{1}{w_{l,m} - w_{l,m+1}} \right), \tag{9}$$

to an equation just above the previous equation, known as q-$\mathrm{P}(E_6^{(1)})$:

$$(\hat{y}z - 1)(\hat{y}\hat{z} - 1) = \frac{(a_1\hat{y} - 1)(a_2\hat{y} - 1)(a_3\hat{y} - 1)(a_4\hat{y} - 1)}{(b_1 q^4 t\hat{y} - 1)(b_2 q^4 t\hat{y} - 1)},$$

$$(yz - 1)(\hat{y}z - 1) = \frac{\theta_1 (z - a_1)(z - a_2)(z - a_3)(z - a_4)}{(b_1 b_2 tz + \theta_1)(a_1 a_2 a_3 a_4 + \theta_1 q^4 tz)},$$

where $\hat{t} = q^4 t$, the a_i, b_i and θ_1 are fixed parameters and $q \in \mathbb{C}$.

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