Quantum Groups and Quantum Cohomology

joint work with Davesh Maulik

Let X be a smooth algebraic variety, say defined by homogeneous polynomial equations

$$F_j(x_0,\dots,x_N) = 0, \qquad j=1,\dots,k$$

in projective space P^N, or a set-theoretic difference of two such.

Quantum cohomology of X is a deformation of its ordinary cohomology that takes into account rational curves in X, e.g. polynomials

$$x_i = x_i(t)$$

of a parameter t that solve $F_j = 0$, modulo reparametrization.

As a linear space with bilinear form, quantum cohomology is the same as the ordinary cohomology with its Poincar\'e form

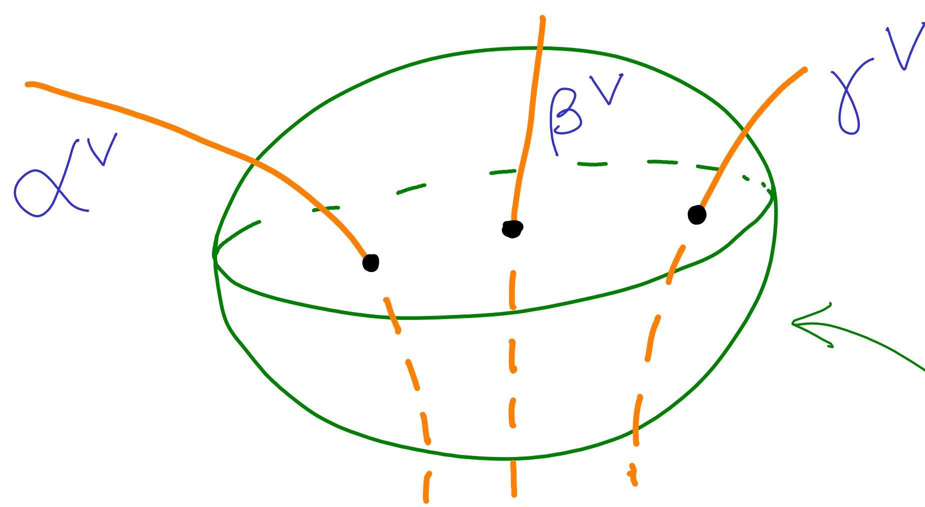
$$(\alpha, \beta) = \int_X \alpha \cup \beta$$

cup product, dual to $\cap$

The only difference is the product. In the quantum case, it involves counts of rational curves in X

$$(\alpha * \beta, \gamma) = \sum q^{\text{degree}} \in \mathbb{Q}[[q]]$$

deformation parameters



rational curves meeting Poincare dual cycles

The degree of a curve is the degree of the polynomials that parametrize it, or more geometrically, its class in H_2(X)

Formally, matrix elements of the quantum product belong to a (completion) of the group algebra of the homology group H_2(X,Z)

$$q^{\text{degree}} \in \mathbb{Q}\left[H_2(X,\mathbb{Z})_{\text{eff}}\right]$$

eff: represented by a holo curve

But in our case the series will converge to a rational function of q.

The symbol q^degree may also be treated as a function on the torus

$$H^2(X,\mathbb{C}) \,/\, 2\pi i\, H^2(X,\mathbb{Z})$$

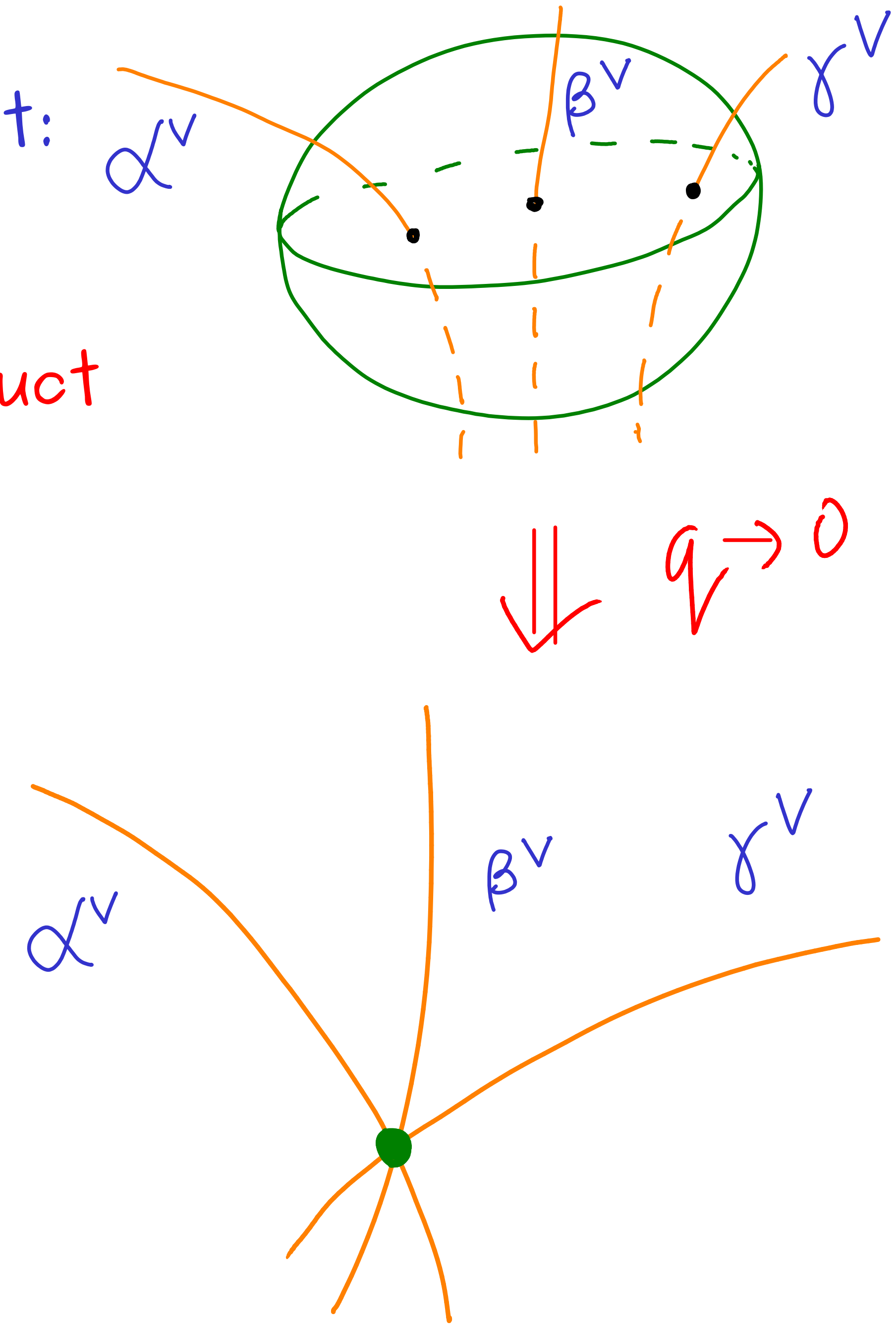
ω ↗

via

$$q^{[\text{Curve}]}(\omega) = \exp\left(-\int_{\text{Curve}} \omega\right)$$

This is natural from the string theory viewpoint.

Basic properties of the quantum product:



(1) * is a deformation of the cup product

As q->0 or \omega >> 0 only the contributions from curves of degree 0 remain. These are points = triple intersections. In other words

$$\alpha * \beta = \alpha \cup \beta + O(q)$$

(2) $*$ is a commutative, associative (!) product
with identity 1 = unit cohomology class

The associativity of the quantum product is a simple, yet remarkable and powerful property of the quantum product.

Beyond these generalities, not much can be said about quantum cohomology of an abstract algebraic variety.

However, for special classes of algebraic varities, quantum cohomology turns out to have very strong ties to quantum integrable systems, and thus can be effectively described.

In particular, quantum integrable systems played a key role in the old work of Givental, Kim, and others on quantum cohomology of homogeneous varieties.

In some sense, the connection with quantum integrable systems is almost tautological.

For any commutative associative product with 1, the operators of multiplication

$$\alpha \star \in \operatorname{End} H^*(X)$$

for a maximal commutative subalgebra and thus a quantum integrable system of sorts.

I think people in the field have a pretty good idea for which X the corresponding quantum integrable system will look like those familiar from quantum groups.

Nekrasov and Shatashvili conjectured such description of quantum cohomology for moduli spaces of vacua in certain SUSY gauge theories.

Bezrukavnikov et al. formulated what turned out to be closely related conjectures for the so-called equivariant symplectic resolution.

Also, there was a similar conjecture by Feigin, Finkelberg, Frenkel, and Rybnikov for Laumon spaces.

Our joint work with Davesh Maulik is about

Nakajima quiver varieties

These are both Nekrasov–Shatashvili vacua and equivariant symplectic resolutions. In fact, they form the largest class of equivariant symplectic resolutions known to date.

There are vere compelling geometric reasons to study quantum cohomology of Nakajima varieties (see below…)

One part of our work, namely

A geometric construction of R-matrices

applies very generally, to all symplectic resolutions and beyond.

It is this part that I will try explain in this talk.

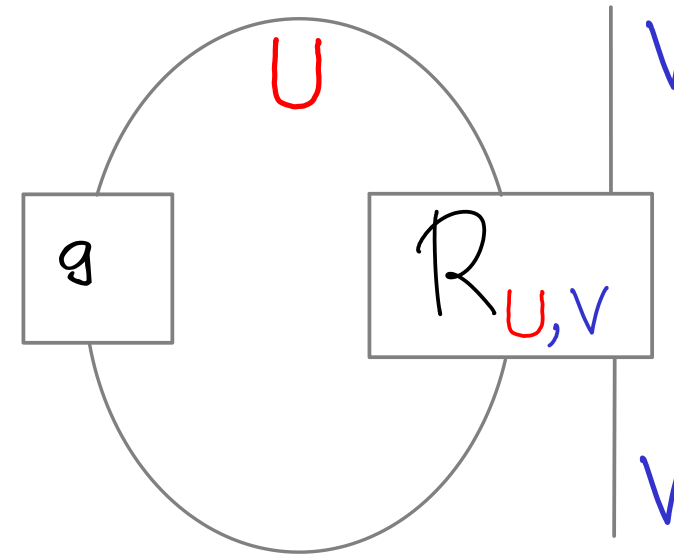
The other part is the actual connection

Baxter subalgebras in Yangians (Commuting transfer matrices) $=$ Operators of quantum mult

I imagine in Australia there is no need to explain the object on the left, but just in case…

If R satisfies the Yang-Baxter equation and g is symmetry then the operators

$$\mathrm{tr}_U (g \otimes 1) R_{U,V}(u)$$



commute for all values of the spectral parameter u and all auxilliary spaces U

The symmetry g of the R-matrices, which has to be fixed here, is identified with the quantum q - a fundamental insight of Nekrasov and Shatashvili. Twist is yet again a regularization.

After this introduction, it may be a good idea to review the definition of Nakajima varieties.

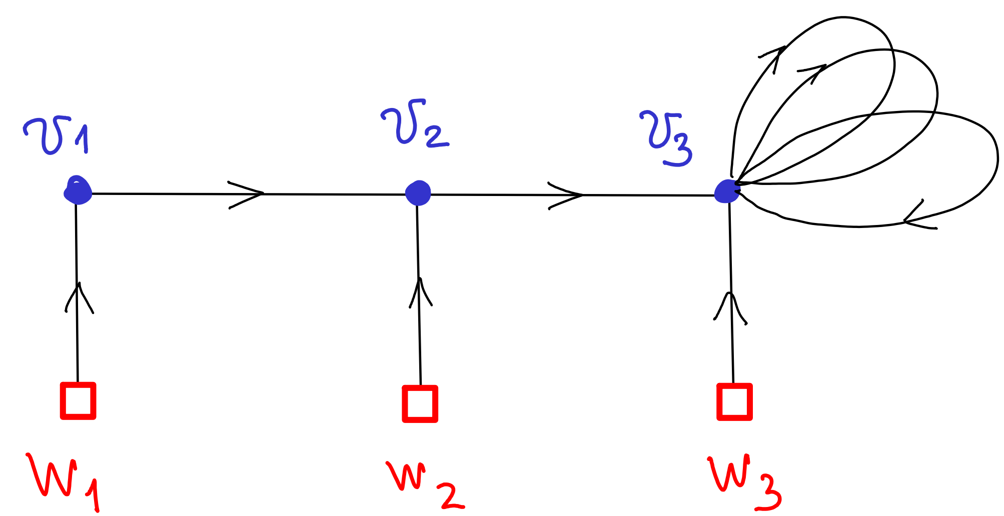
Nakajima varieties $M(v,W)$ are algebraic Hamiltonian reductions of

T^* Framed representations of a quiver Q

with dimension vectors v and w by the action of $GL(v) = \prod GL(v_i)$

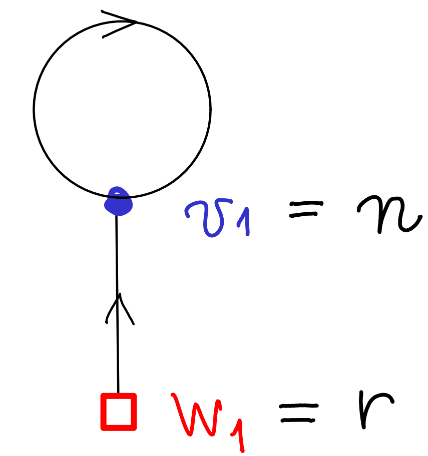
original vertices →

framing vertices →



The reduction depends on parameters $(\theta, \zeta) \in \mathfrak{z}(\check{\mathfrak{u}}(v)) \otimes$ imaginary quaternions

For example, for the quiver with one vertex and one loop, the Nakajima variety is



M(r,n) = moduli of framed U(r)-instantons on R^4 of charge n

or, rather, the partial compactification of the RHS that parametrizes framed torsion-free sheaves of rank r on C^2

This is, obviously, an object of central importance in 4D gauge theories.

For the same quiver without the loop, one gets T^*Gr(n,r), the cotagent bundle of the Grassmannian of n-planes in an r-space.

It is natural and convenient to form disconnected Nakajima varieties

$$\mathcal{M}(w) = \coprod_{v} \mathcal{M}(v,w)$$

This means summation over instanton charges for framed instantons or subspace dimension for Grassmannians.

In the work of Nakajima and Varagnolo a certain Yangian was shown to act on the cohomology of M(w). This action will be recovered and extended in our approach.

Connected components are the weight spaces for its Cartan subalgebra.

As Yangian modules,

$$H^*(\mathcal{M}(w_1+w_2)) = H^*(\mathcal{M}(w_1)) \otimes H^*(\mathcal{M}(w_2))$$

which may be traced to the natural embedding

$$\mathcal{M}(w_1) \times \mathcal{M}(w_2) \hookrightarrow \mathcal{M}(w_1+w_2)$$

by direct sum of quiver representations.

For us, it is crucial here that the LHS is the fixed locus of a Hamiltonian C^*-action on the RHS. This C^* acts by changing the framing. E.g. for framed instantons by constant gauge trasformations (in which case LHS is formed by reducible connections).

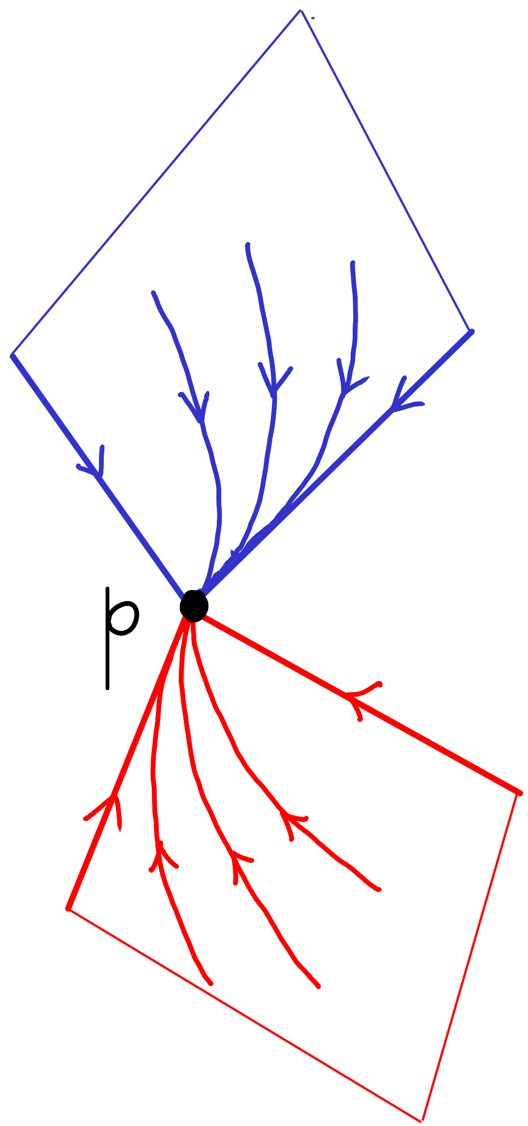
For any algebraic Hamiltonian action of a torus A on a symplectic variety, we define a "wrong way" map

$$\mathrm{Stab}_C : H^*(X^A) \longrightarrow H^*_A(X)$$

to the A-equivariant cohomology of X.

In this talk, let's make a simplifying assumption that the fixed-point set X^A is finite. Then Stab(p), where p is a fixed point, is an A-invariant middle-dimensional (in fact, Lagrangian) subvariety of X.

It depends on an additional discrete choice – a choice of a chamber C in Lie algebra of A.



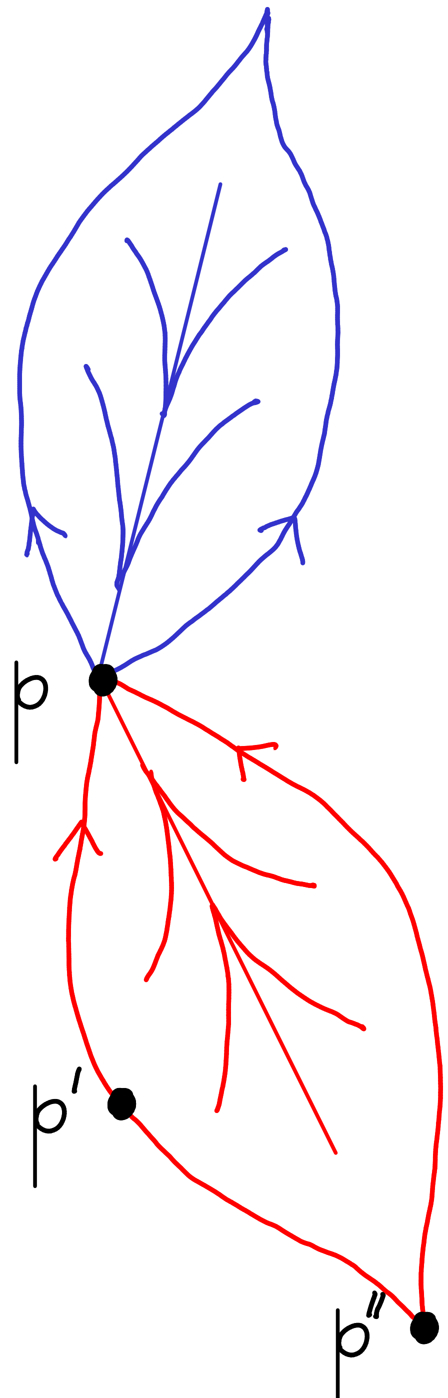
Let p be a fixed point of A and let ξ be in the Lie algebra of A.

The tangent space at p splits into stable and unstable directions with respect to ξ - those attracted and repelled from p under the action of

exp(t*ξ)

as t->-\infty.

These jump as ξ crosses hyperplanes defined by the weights α of A in the tangent space at p.



Globally on X, these become the stable and unstable leaves of p – the closures of the attracting/repelling manifolds of p.

These also jump as ξ\in Lie A crosses walls.

This gives a partial order on X^A, namely

$$p' \leq p \iff p' \in \mathrm{Leaf}(p)$$

Just like real leaves, the stable leaves are typically quite singular at the other fixed points p'.

Example: (conormals to) Schubert varieties

By definition,

$$\mathrm{Stab}(p) = \mathrm{Leaf}(p) + \sum_{p' < p} m_{p,p'}\, \mathrm{Leaf}(p')$$

where the coefficients m_{p,p'} are uniquely determined by

$$\mathrm{degree}_A\, \mathrm{Stab}(p)\Big|_{p'} < \frac{1}{2}\, \mathrm{codim}\, p'$$

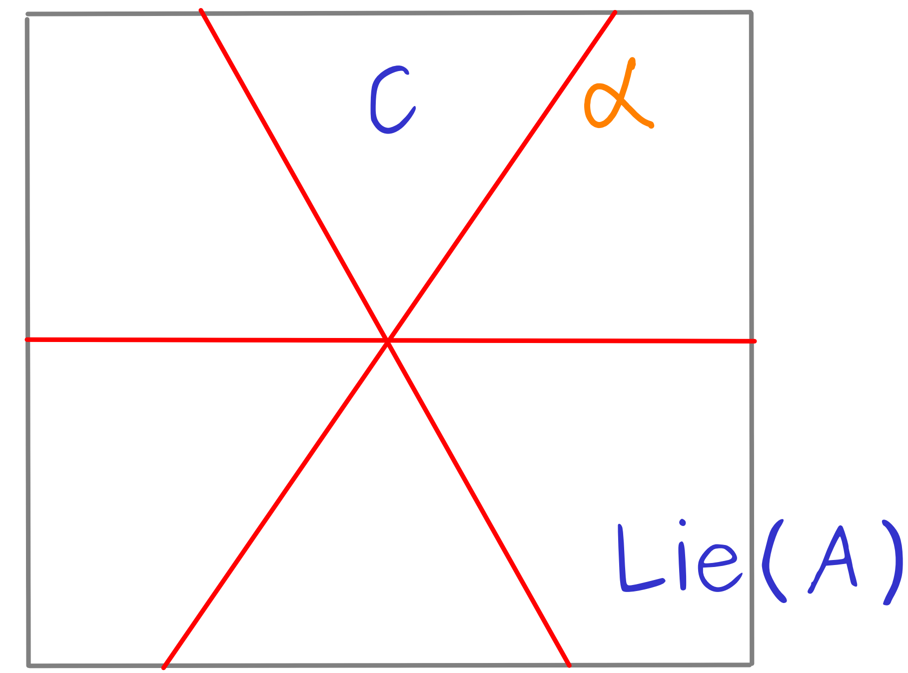
$$\mathrm{codim}\, p' = \dim X$$

Suitably interpreted, also works in K-theory.

The idea is that while leaves are delicate and difficult to control the stable envelopes are much more robust and more suitable for computations.

In fact, when the computations are set up properly, one never has to think about leaves.

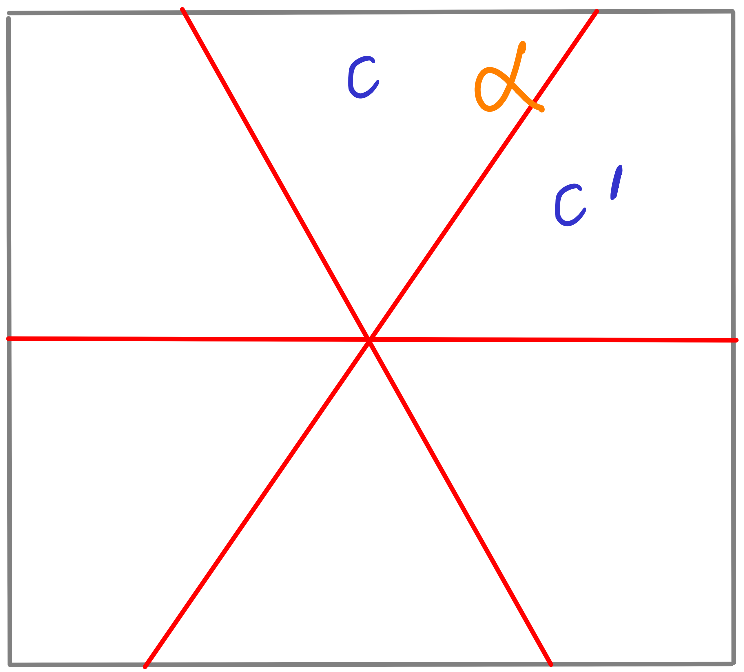
The leaves and, hence, the stable envelopes jump as the Lie algebra parameter ξ crosses walls formed by weights α of A in the tangent spaces at fixed points.



For every chamber C of the complement of the walls, we get a canonical map

$$\mathrm{Stab}_C : H^*_G(X^A) \longrightarrow H^*_G(X)$$

where G is any group that commutes with A.



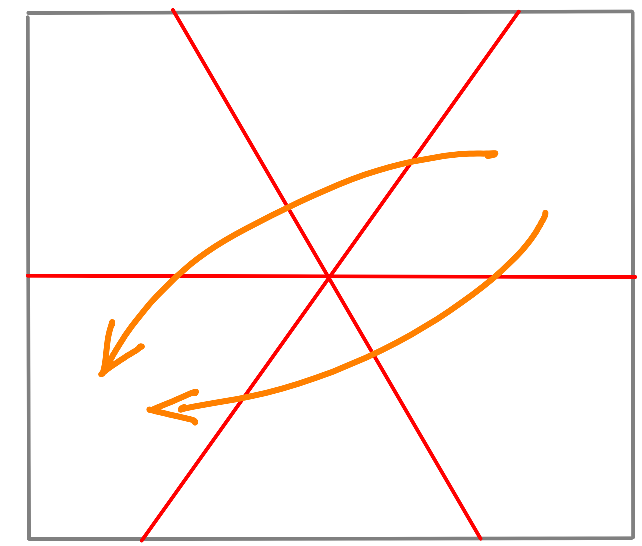
If we cross a wall **α**, we get a map

$$R_\alpha = \mathrm{Stab}_{C'}^{-1} \circ \mathrm{Stab}_{C}$$

that acts in the cohomology of X^A and depends rationally on the equivariant parameters.

The weight **α** is the spectral parameter for R_**α**

These R-matrices satisfy natural compatibilities that translate into the Yang-Baxter equation for the case of



$$\mathcal{M}(v,w) \times \mathcal{M}(v',w') \hookrightarrow \mathcal{M}(v+v',w+w')$$

Once there is a solution R of the Yang-Baxter equation, we can define a Yangian action on the cohomology. This is done FRT-style, as the algebra generated by the matrix coefficients of $R(u)$ in one of the factors. By construction, they act in the other factor.

We show this is indeed a Yangian, that is, a Hopf algebra deformation of $U(\mathfrak{g}[t])$ for a certain Lie algebra $\mathfrak{g}$.
In general, $\mathfrak{g}$ goes beyond Kac-Moody Lie algebras.

For quivers of finite type, this is the same Yangian
as in the work of Nakajima and Varagnolo. In general,
it is larger.

For example, for the case of instantons on R^4, the
R-matrix is essentially the reflection operator in Liouville CFT.
Algebras closely related to the corresponding Yangian appeared
in the work of Cherednik, Miki, Vasserot-Schiffmann, and many others.

This directly relates to many topics of current research,
in particular to the work of Alday, Gaiotto, and Tachikawa,
and also many other people.

The Cartan subalgebra h of g acts by linear functions of v and w.

The subalgebra U(h[t]) of U(g[t]) is deformed to the algebra of cup products by the characteristic classes of the tautological bundles.

It also corresponds to the vacuum (that is, v=0) matrix elements of the R-matrix and thus is the g->0 limit of the Baxter subalgebras

$$\mathrm{tr}_{U}\,(g\otimes 1)\,R_{U,V}(u)$$

in the Yangian. The general g gives quantum products, as predicted by [NS]

In particular, we prove the following formula for the quantum product by first Chern classes λ of tautological bundles

$$\lambda \star = \lambda \cup + \hbar \sum_{\alpha > 0} (\lambda, \alpha) \frac{q^\alpha}{1 - q^\alpha} e_\alpha e_{-\alpha} + \dots$$

scalar

where

\hbar = equivariant weight of the symplectic form,

α ranges over the roots of g, which form a subset of H_2(X,Z),

e_α and e_{$-\alpha$} are dual bases of the corresponding root spaces,

and we have shifted the origin in H^2(X) to get rid of many signs.

It proves roughly half of what Bezrukavnikov et al conjectured.

Namely it equates the so-called quantum differential equation

$$\frac{d}{d\lambda} \Psi(q) = \lambda \star \Psi(q), \qquad \Psi \in H^*(X), \quad \lambda \in H^2(X)$$

with the trigonometric Casimir connection for $\mathfrak{g}$

Trigonometric Casimir connections for the (Yangians of) finite-dimensional semisimple Lie algebras $\mathfrak{g}$ were defined and studied by Toledano Laredo (building on the earlier work of ...)

This is proven by geometrically defining and identifying the commuting difference Quantum Knizhnik-Zamolodchikov connection on the space of equivariant parameters.

In other words, there is a flat differential-difference connection on the whole equivariant $H^2_G(X)$, which includes both geometric and purely equivariant directions. Cocharacters of G produce nontrivial X bundles over P^1. The new operators count sections of these bundles, i.e. twisted rational curves in X.

In the language of quantum integrable system, the commuting difference connection gives shift operators.

Now a few words about applications and extensions ...

Quantum cohomology of Nakajima varieties is interesting, in particular, because counting curves in Nakajima varieties is related to questions in Donaldson–Thomas theory.

DT theory is the analog of Donaldson theory in 3 complex (6 real) dimensions. Instead of bundles and sheaves on Kahler surfaces, it counts sheaves on algebraic 3-folds. (As well as stable objects in similar categories).

It is directly related to many things in topological strings, e.g. the GW=DT conjectures of [MNOP] equate it (after a change of variables), to the A model on the same 3-fold.

Sheaves on fibrations like

$$\text{surface } S \hookrightarrow \text{3-fold}$$
$$\downarrow$$
$$\mathbb{P}^1$$

are related to rational curves in the moduli of sheaves on S.

Thus quantum cohomology of Nakajima varieties provides what may be called a Hamiltonian viewpoint on the DT theory.

Among other things, our results give a very different proof of the old results of [O-Pandharipande] and [Oblomkov-Maulik] on quantum cohomology of Hilbert schemes of A_n sufaces (on which GW=DT for toric varieties rests), as well its generalization for all ranks.

In some sense, general rank is easier than rank 1 !

The generalization of this to K-theory and beyond
is the subject of current research.

In K-theory, the picture is more complete and symmetric,
with the quantum connection and shift operators playing
symmetric roles.

This symmetry is perhaps best explained from the
viewpoint of M-theory.

Such a close connection between quantum integrable systems and the geometry of extended objects in extra dimensions brings many philosophical questions, like ...