Eynard-Orantin invariants and Cohomological field theories

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Paul Norbury

Melbourne

Outline

- Eynard-Orantin invariants
- Cohomological field theories
- Givental's viewpoint of CohFT
- Geometric relation between Givental and Eynard-Orantin

Eynard-Orantin invariants

$$\begin{aligned} \textit{Input}: &\ \text{Riemann surface } C, \text{ meromorphic } x, y : C \to \mathbb{C} \\ &\ dx \text{ has simple zeros, disjoint from zeros of } dy. \\ \textit{Output}: &\ \text{meromorphic multidifferentials } \omega^g_n(p_1, \ldots, p_n),\ p_i \in C \end{aligned}$$

Defined recursively: $\omega^0_1 = ydx$

$$\delta\omega^g_n(p_1, \ldots, p_n) = \int_{\gamma(p)} \Lambda(p)\omega^g_{n+1}(p, p_1, \ldots, p_n) \Leftarrow \text{variation of } (C, x, y)$$

$\omega^g_n(p_1, \ldots, p_n)$ are generating functions for enumerative problems.

1. $x = y^2$—intersection numbers on $\overline{\mathcal{M}}_{g,n}$, EO, K.
2. $x = \log y - y$—simple Hurwitz numbers BEMS, ELSV.
3. $xy - y^2 = 1$—Belyi Hurwitz problem, N, K.
4. $x = 2\cosh y$—stationary GW invariants of $\mathbb{P}^1$, NS/DOSS, OP.

AIM: Find context for Eynard-Orantin invariants

Theorem (Dunin-Barkowski, Orantin, Shadrin, Spitz 2012)

Semi-simple cohomological field theories are equivalent to Eynard-Orantin invariants of local curves.

Theorem (N., Scott $g = 0, 1$, DOSS $g > 1$)

For the curve $x = z + 1/z, y = \ln z$ an analytic expansion of $\omega_n^g(p_1, ..., p_n)$ around a branch of $x_i = \infty$ is

$$\Omega_n^g(x_1, ..., x_n) = \sum_{\mathbf{b}} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_g^{\mathbb{P}^1} \cdot \prod_{i=1}^{n} \frac{(b_i + 1)!}{x_i^{b_i+2}} dx_i .$$

NS and DOSS proofs are indirect.

Cohomological field theories

- Vector space (H,η) and a sequence of S_n-equivariant maps

$$I_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

which satisfy compatibility conditions from inclusion of strata:

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

- Examples:
  - $I_{g,n} \in H^0(\overline{\mathcal{M}}_{g,n}) \Rightarrow$ 2D topological field theories
  - $\dim H = 1 \Rightarrow I_{g,n}$ determined by an element of $H^*(\overline{\mathcal{M}}_{g,n})$
  - Gromov-Witten invariants (lose information ... but equivalent)
    - $\overline{\mathcal{M}}_{g,n}(X,d) = \{\pi : (\Sigma, p_1, \ldots, p_n) \overset{\text{degree } d}{\longrightarrow} X\}/\sim$
    - $I_{g,n} : H^{\otimes n} = H^*(X)^{\otimes n} \xrightarrow{ev^*} H^*(\overline{\mathcal{M}}_{g,n}(X,d)) \to H^*(\overline{\mathcal{M}}_{g,n})$
- Correlators: $k_i \in \mathbb{N}$, $e_\nu \in H$, $\nu = 0, \ldots, N-1$

$$\langle \tau_{k_1}(e_{\nu_1}) \ldots \tau_{k_n}(e_{\nu_n})\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(e_{\nu_1}, \ldots, e_{\nu_n}) \cdot \prod_{j=1}^n \psi_j^{k_j}$$

Partition function

- $F_g(\{t_k^\nu\}) = \left\langle \exp \sum_{k=0}^{\infty} \tau_k(t_k) \right\rangle_g$ for $t_k = t_k^\nu e_\nu$
- Examples
  - $F_0 = \frac{t_0^3}{6} + \frac{t_0^3 t_1}{6} + \frac{t_0^3 t_1^2}{6} + \frac{t_0^4 t_2}{24} + \frac{t_0^4 t_1 t_2}{8} + \dots \quad (t_k = t_k^0)$
  - $F_0 = \frac{(t_0^0)^2 t_0^1}{2} + \exp t_0^1 + \frac{(t_1^1)^2}{2} + \frac{t_2^1}{4} + \dots$
- Properties:

$$\text{(DE)} \qquad \partial_{0,1} F_0 = -2F_0 + \sum_{k=0}^{\infty} t_k^\nu \partial_{\nu,k} F_0$$

$$\text{(SE)} \qquad \partial_{0,0} F_0 = \frac{1}{2} t_0^\mu t_0^\nu \eta_{\mu\nu} + \sum_{k=0}^{\infty} t_{k+1}^\nu \partial_{\nu,k} F_0$$

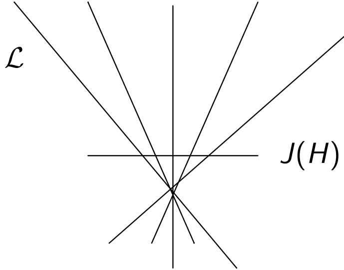
$$\text{(TRR)} \quad \partial_{\alpha,k+1} \partial_{\beta,l} \partial_{\gamma,m} F_0 = \partial_{\alpha,k} \partial_{\mu,0} F_0 \cdot \eta^{\mu\nu} \cdot \partial_{\beta,l} \partial_{\gamma,m} \partial_{\nu,0} F_0$$

Givental's viewpoint

Represent genus 0 partition function geometrically.

$$H((\hbar,\hbar^{-1})) \supset \mathcal{L} = \{t_0 + t_1\hbar + \ldots + \frac{\partial F_0}{\partial t_0}\hbar^{-1} + \frac{\partial F_0}{\partial t_1}\hbar^{-2} + \ldots\}$$

$$(t_k = t_k^\nu e_\nu)$$



$\hbar T\mathcal{L} \subset T\mathcal{L} \Leftrightarrow F_0$ satisfies SE, DE, TRR

$\mathcal{L}$ determined by a section, the J-function $J : H \to \mathcal{L}$

Geometric relation between Givental and Eynard-Orantin

Follow K. Saito, Dubrovin, Barannikov

- (C, x), affine curve and holomorphic function $x : C \to \mathbb{C}$
- Milnor ring $R_x \cong \mathbb{C}^{|D|}$ where $dx^{-1}(0) = D \subset C$
  - any R-module N defines a quotient ring $R \to R/\mathrm{Ann}\ N$
  - R_x defined by the $\mathbb{C}[C]$-module $H^0(K_C)/\langle dx\rangle$
- space of deformations $\{(C_t, x_t), t \in H\}$ with $\begin{array}{rcl} T_t H & \stackrel{\cong}{\to} & R_{x_t} \\ v & \mapsto & \partial_v x_t \end{array}$

Examples:

- C rational
  - $x_t(z) = z^d + t_{d-2}z^{d-2} + \ldots + t_0$, $R_{x_t} = \mathbb{C}[z]/x_t'(z)$ (Sabbah)
  - $x_t(z) = (z^d + t_{d-1}z^{d-1} + \ldots + t_0)/z^k$, $R_{x_t} = \mathbb{C}[z, z^{-1}]/x_t'(z)$
- C elliptic, $x = \wp(z)$, Weierstrass $\wp$-function

$$x = \wp(z), \quad R_\wp = \mathbb{C}[\wp]/(-4\wp^3 + a\wp + b)$$

$\dim H = 3 \Rightarrow$ must deform x *and* C

- metric on $H \cong R_x$ defined by differential dy on C, (no zeros in common with dx)

$$\langle p, q \rangle := \operatorname*{Res}_{x=\infty} \frac{pdy \cdot qdy}{dx}, \quad p, q \in R_x$$

- locally free sheaf E over $H \times \mathbb{C}^\times$
  - E_t = sheaf of 1-forms over C_t
  - flat connection ∇^{GM} depending on h
-

$$\begin{aligned} H((\hbar, \hbar^{-1})) &= \{s \in \Gamma(M, E \otimes_{\mathcal{O}_M((h))} \mathcal{O}_M((h, h^{-1}))|\nabla s = 0\} \\ &\cong T_t H \otimes \mathbb{C}((\hbar, \hbar^{-1})) \end{aligned}$$

- $\mathcal{L} \subset H((\hbar, \hbar^{-1}))$
  - ruled by embeddings $E_t \subset H((\hbar, \hbar^{-1}))$

Conclusion

- Givental: $\mathcal{L} \subset H((\hbar, \hbar^{-1}))$
- Eynard-Orantin: (C, x, y)
- Constructed $\mathcal{L} \subset H((\hbar, \hbar^{-1}))$ from (C, x, y)
- Need to prove that this contruction realises DOSS
- Proven for $GW(pt)$ and $GW(\mathbb{P}^1)$