

The Temperley-Lieb algebra  
○○○○

The link representations  
○○

The XXZ representation  
○○○○○

Jordan structure  
○

# *The Jordan structure of periodic loop models*

Alexi Morin-Duchesne

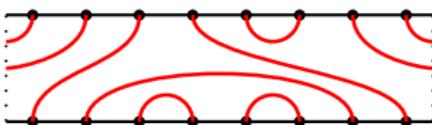
Centre de Recherches Mathématiques, Université de Montréal

ANZAMP Meeting, Lorne,  
December 3, 2012

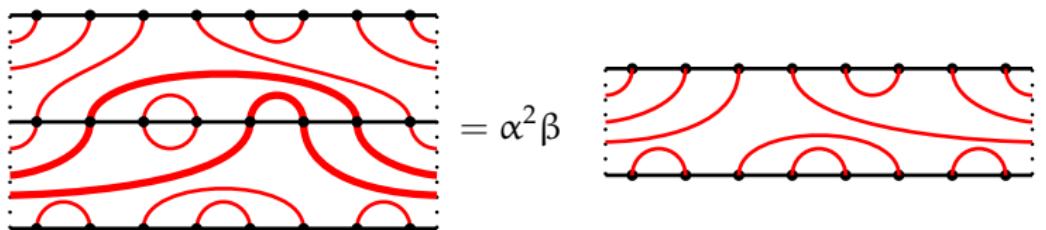
*Joint work with Yvan Saint-Aubin*

# The periodic Temperley-Lieb algebra $TLP_N(\alpha, \beta)$

A connectivity is a set of **non-intersecting curves** connecting  $2N$  nodes,  $N$  on the top and  $N$  on the bottom of a periodic strip.



The product between connectivities is given by



$TLP_N(\alpha, \beta)$  is the vector space generated by connectivities and endowed with this product.

# The algebra $TLP_N(\alpha, \beta)$

The Temperley-Lieb algebra can be generated by

$$\begin{aligned}
 id &= \text{ (Diagram: } N \text{ points on a horizontal line, red vertical strands connecting } 1 \text{ to } 1, 2 \text{ to } 2, \dots, N \text{ to } N) \\
 e_i &= \text{ (Diagram: } N \text{ points, red strands from } 1 \text{ to } i-1, i \text{ to } i, i+1 \text{ to } N) \\
 \Omega &= \text{ (Diagram: } N \text{ points, red strands from } 1 \text{ to } 2, 2 \text{ to } 3, \dots, N-1 \text{ to } N) \\
 e_0 &= \text{ (Diagram: } N \text{ points, red strands from } 1 \text{ to } 2, 2 \text{ to } 1, 3 \text{ to } 4, \dots, N-1 \text{ to } N) \\
 \Omega^{-1} &= \text{ (Diagram: } N \text{ points, red strands from } 1 \text{ to } 2, 2 \text{ to } 1, 3 \text{ to } 4, \dots, N-2 \text{ to } N, N-1 \text{ to } N)
 \end{aligned}$$

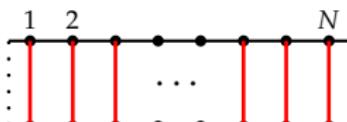
Any connectivity is obtained by a multiplication of some generators.

$$e_0e_1e_3e_2 = \text{ (Diagram: } N \text{ points, red strands forming a complex web of connections)} = \text{ (Simpler diagram: } N \text{ points, red strands forming a simplified web of connections)}$$

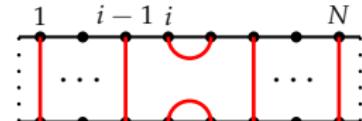
# The algebra $TLP_N(\alpha, \beta)$

The Temperley-Lieb algebra can be generated by

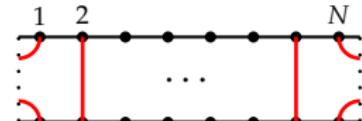
$$id =$$



$$e_i =$$



$$e_0 =$$



$$\Omega =$$

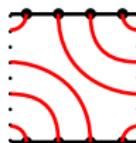


$$\Omega^{-1} =$$



Curves can wind around the cylinder indefinitely, so  $TLP_N(\alpha, \beta)$  is infinite dimensional!

$$e_0 e_1 e_2 e_3 e_0 =$$



# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$e_N = e_0$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$

# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

$$e_N = e_0$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$

$$(e_2)^2 = \begin{array}{c} \text{Diagram showing two strands crossing twice, resulting in a loop. Red lines highlight the strands.} \\ \vdots \end{array} = \beta \begin{array}{c} \text{Diagram showing a single loop with two strands crossing it. Red lines highlight the strands.} \\ \vdots \end{array} = \beta e_2$$

# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

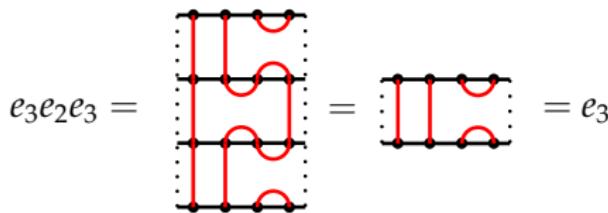
$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$e_N = e_0$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$



# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

$$e_N = e_0$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$

$$e_3 e_1 = \begin{array}{c} \text{Diagram showing two strands from top-left to bottom-right, with a red loop connecting them.} \\ \text{A dashed box encloses the strands and loop.} \end{array} = \begin{array}{c} \text{Diagram showing two strands from top-left to bottom-right, with red loops on both strands.} \\ \text{A dashed box encloses the strands and loops.} \end{array} = e_1 e_3$$

# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

$$e_N = e_0$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

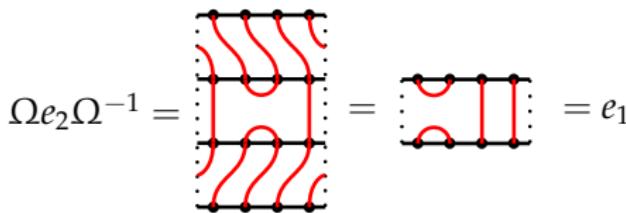
$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$



# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

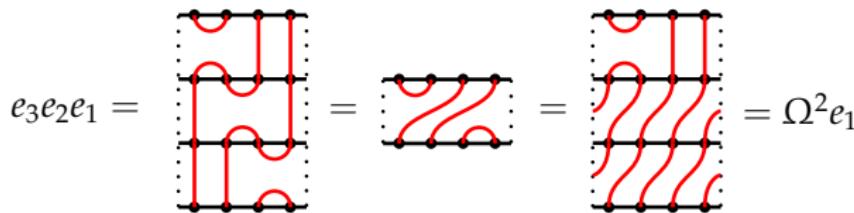
$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$e_N = e_0$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$



# The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$

$$e_i e_{i \pm 1} e_i = e_i$$

$$e_i e_j = e_j e_i \quad (|i - j| > 1)$$

$$e_N = e_0$$

$$\Omega \Omega^{-1} = \Omega^{-1} \Omega = id$$

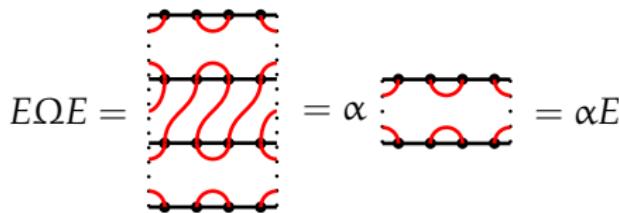
$$\Omega e_i \Omega^{-1} = e_{i-1},$$

$$e_{N-1} e_{N-2} \dots e_2 e_1 = \Omega^2 e_1$$

$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$

$$E \Omega^{\pm 1} E = \alpha E$$

$$(where E = e_0 e_2 e_4 \dots e_{N-2})$$



# The transfer matrix and Hamiltonian

The **transfer matrix** and **Hamiltonian** are elements of  $TLP_N(\alpha, \beta)$ .

Transfer matrix:

$$T_N(\lambda, \nu) = \overbrace{\begin{array}{c|c|c|c} \bullet & \bullet & \dots & \bullet \\ \nu & \nu & & \nu \\ \bullet & \bullet & \dots & \bullet \end{array}}^N$$

where =  $\sin(\lambda - \nu)$  +  $\sin \nu$

Hamiltonian:

$$\mathcal{H} = \sum_{i=0}^{N-1} e_i = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4}.$$

# Link states and the representation $\rho$

A representation of  $TLP_N(\alpha, \beta)$  is obtained by defining the link states and the action of connectivities on link states:

$$B_4^0 = \{ \text{Diagram 1}, \text{Diagram 2}, \text{Diagram 3}, \text{Diagram 4}, \text{Diagram 5}, \text{Diagram 6}, \text{Diagram 7}, \text{Diagram 8} \},$$

$$B_4^2 = \{ \text{Diagram 9}, \text{Diagram 10}, \text{Diagram 11}, \text{Diagram 12}, \text{Diagram 13} \}, \quad B_4^4 = \{ \text{Diagram 14} \}.$$

The action of  $TLP_N(\alpha, \beta)$  elements on link states:

$$\text{Diagram 15} = \beta \text{ Diagram 16} \quad \rho\left(\text{Diagram 17}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^8 & \beta v^6 & v^8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

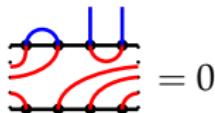
# Link states and the representation $\rho$

A representation of  $TLP_N(\alpha, \beta)$  is obtained by defining the link states and the action of connectivities on link states:

$$B_4^0 = \{ \text{link states with 0 loops}, \text{link states with 1 loop}, \text{link states with 2 loops}, \text{link states with 3 loops}, \text{link states with 4 loops} \},$$

$$B_4^2 = \{ \text{link states with 2 vertical lines}, \text{link states with 1 vertical line and 1 loop}, \text{link states with 2 loops and 1 vertical line} \}, \quad B_4^4 = \{ \text{link states with 4 vertical lines} \}.$$

The action of  $TLP_N(\alpha, \beta)$  elements on link states:



$$\rho \left( \text{link state with 2 loops and 1 vertical line} \right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 & \alpha & 1 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & v^8 & \beta v^6 & v^8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$d=0$        $d=2$        $d=4$

# Link states and the representation $\rho$

A representation of  $TLP_N(\alpha, \beta)$  is obtained by defining the link states and the action of connectivities on link states:

$$B_4^0 = \{ \text{link states with 0 loops}, \text{link states with 1 loop}, \text{link states with 2 loops}, \text{link states with 3 loops}, \text{link states with 4 loops} \},$$

$$B_4^2 = \{ \text{link states with 2 vertical lines}, \text{link states with 1 vertical line and 1 loop}, \text{link states with 2 loops and 1 vertical line} \}, \quad B_4^4 = \{ \text{link states with 4 vertical lines} \}.$$

The action of  $TLP_N(\alpha, \beta)$  elements on link states:

$$\rho \left( \text{link state with strands} \right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v^8 & \beta v^6 & v^8 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$d=0$        $d=2$        $d=4$

$v$  is the **twist parameter**.

# The Hamiltonian

$\mathcal{H}$  in the link representation:

$$\rho(\mathcal{H}) = \left( \begin{array}{cccccc|cccccc|c} 2\beta & 2 & \alpha & 0 & \alpha & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 2 & 2\beta & 2 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \beta & v^2 & 0 & v^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & \beta & v^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & \beta & v^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & v^{-2} & \beta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

For specific values of  $\alpha$ ,  $\beta$  and  $v$   
 $\mathcal{H}$  is **non diagonalizable!**

For example:  $\beta = 0$ ,  $\alpha = 2$ ,  
 a rank 2 Jordan cell  
 appears in the  $d = 0$  sector.

$$Sp(\mathcal{H})S^{-1} = \left( \begin{array}{cccc|ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

**Jordan cells** for finite  $N$  → **Logarithmic CFT** in the scaling limit

# The XXZ Hamiltonian

The generalized Hamiltonian is given by

$$H_{XXZ} = \frac{1}{2} \sum_{j=0}^{N-1} \left( \frac{v^2 + v^{-2}}{2} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) - \frac{v^2 - v^{-2}}{2i} (\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x) + \frac{u^2 + u^{-2}}{2} (\sigma_j^z \sigma_{j+1}^z - id) \right)$$

- $\sigma_j^a = \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{j-1} \otimes \sigma^a \otimes \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{N-j}$ .
- $\sigma_N^a \equiv \sigma_0^a$ .
- It acts on  $(\mathbb{C}^2)^{\otimes N}$ . For  $N = 4$ :  $|\downarrow\downarrow\downarrow\downarrow\rangle, |\downarrow\downarrow\downarrow\uparrow\rangle, |\downarrow\downarrow\uparrow\downarrow\rangle, \dots, |\uparrow\uparrow\uparrow\uparrow\rangle$ .
- The usual case is just  $v^2 = 1$  and  $\Delta = \frac{u^2 + u^{-2}}{2}$ .
- $H_{XXZ}$  is diagonalizable for  $u = e^{i\phi}, v = e^{i\gamma}$  and  $\gamma, \phi \in \mathbb{R}$ .

# The XXZ representation of $TLP_N$

It can be rewritten as

$$H_{XXZ} = \sum_{j=0}^{N-1} \bar{e}_j, \quad \text{with}$$

$$\bar{e}_j = \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u^2 & v^2 & 0 \\ 0 & v^{-2} & u^{-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{N-j-1}$$

- $H_{XXZ}$  commutes with  $S^z = \frac{1}{2} \sum_{j=0}^{N-1} \sigma_j^z$ .
- Let  $t$  be the operator that translates all spins one position to the left. The matrices  $\bar{e}_j$ , along with  $\bar{\Omega} = v^{2S^z} t$  satisfy all the relations of  $TLP_N(\alpha, \beta)$ , for  $\beta = u^2 + u^{-2}$ , and  $\alpha = v^N + v^{-N}$  for  $N$  even.

# The XXZ Hamiltonian: an example for $N = 4$

In the spin basis  $\{ \dots, |\uparrow\uparrow\downarrow\downarrow\rangle, |\uparrow\downarrow\uparrow\downarrow\rangle, |\uparrow\downarrow\downarrow\uparrow\rangle, |\downarrow\uparrow\uparrow\downarrow\rangle, |\downarrow\uparrow\downarrow\uparrow\rangle, |\downarrow\downarrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\uparrow\uparrow\rangle, |\uparrow\uparrow\uparrow\downarrow\rangle, |\uparrow\uparrow\downarrow\uparrow\uparrow\rangle, |\uparrow\downarrow\uparrow\uparrow\uparrow\rangle, |\downarrow\uparrow\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\uparrow\uparrow\uparrow\rangle \}$ ,

$$\left( \begin{array}{cccc|cccc|cccc} \ddots & \vdots \\ \dots & u^2 + u^{-2} & v^2 & 0 & 0 & v^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & v^{-2} & 2(u^2 + u^{-2}) & v^2 & v^2 & v^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & 0 & v^{-2} & u^2 + u^{-2} & 0 & v^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & 0 & v^{-2} & 0 & u^2 + u^{-2} & v^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & v^2 & 0 & v^{-2} & v^{-2} & 2(u^2 + u^{-2}) & v^2 & 0 & 0 & 0 & 0 & 0 \\ \dots & 0 & v^2 & 0 & 0 & v^{-2} & u^2 + u^{-2} & 0 & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 & 0 & u^2 + u^{-2} & v^2 & 0 & v^{-2} & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & u^2 + u^{-2} & v^2 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & u^2 + u^{-2} & v^2 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & v^{-2} & u^2 + u^{-2} & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

# The XXZ Hamiltonian: an example for $N = 4$

Setting  $u = e^{i\pi/4}$  ( $\beta = 0$ ) and  $v = 1$  ( $\alpha = 2$ ), it can be diagonalized:

$$\bar{S}H_{XXZ}\bar{S}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \textcolor{red}{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the same as  $S\rho(H)S^{-1}$  except for the Jordan block!

# The map $i_N^d$

$i_N^d$ : link states with  $n$  bubbles  $\rightarrow$  spin states with  $n$  down arrows.

$$i_4^4 (\text{Diagram: four vertical lines with dots at top and bottom}) = |\uparrow\uparrow\uparrow\uparrow\rangle,$$

$$\begin{aligned} i_4^2 (\text{Diagram: four vertical lines with a bubble between second and third}) &= (uv\sigma_3^- + u^{-1}v^{-1}\sigma_2^-)|\uparrow\uparrow\uparrow\uparrow\rangle \\ &= uv|\uparrow\uparrow\downarrow\uparrow\rangle + (uv)^{-1}|\uparrow\downarrow\uparrow\uparrow\rangle, \end{aligned}$$

$$\begin{aligned} i_4^0 (\text{Diagram: two bubbles}) &= (uv\sigma_2^- + u^{-1}v^{-1}\sigma_1^-)(uv\sigma_4^- + u^{-1}v^{-1}\sigma_3^-)|\uparrow\uparrow\uparrow\uparrow\rangle \\ &= (uv)^2|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + (uv)^{-2}|\downarrow\uparrow\downarrow\uparrow\rangle, \end{aligned}$$

$$\begin{aligned} i_4^0 (\text{Diagram: three bubbles}) &= (uv^3\sigma_4^- + u^{-1}v^{-3}\sigma_1^-)(uv\sigma_3^- + u^{-1}v^{-1}\sigma_2^-)|\uparrow\uparrow\uparrow\uparrow\rangle \\ &= u^2v^4|\uparrow\uparrow\downarrow\downarrow\rangle + v^2|\uparrow\downarrow\uparrow\downarrow\rangle + v^{-2}|\downarrow\uparrow\downarrow\uparrow\rangle + u^{-2}v^{-4}|\downarrow\downarrow\uparrow\uparrow\rangle. \end{aligned}$$

# The transformation $i_N^d$

## *Proposition 1*

$i_N^d$  is a **homomorphism** if  $\beta = u^2 + u^{-2}$  and  $\alpha = v^N + v^{-N}$ :  
 $i_N^d(e_i v) = \bar{e}_i i_N^d(v)$  and  $i_N^d(\Omega v) = \bar{\Omega} i_N^d(v)$  for any link state  $v$ .

## *Proposition 2*

$i_N^d$  is an **isomorphism** except if  $u$  and  $v$  are such that

$$\prod_{k=1}^{(N-d)/2} \left( (iu)^{4k+2d} v^{2N} - 1 \right) = 0$$

- $\rho(\mathcal{H})$  and  $H_{XXZ}$  have the same spectrum if  $\beta = u^2 + u^{-2}$  and  $\alpha = v^N + v^{-N}$ .
- When  $i_N^d$  is an isomorphism,  $\rho(\mathcal{H})$  is **diagonalizable**.

# Jordan cells

(1,1)

(2,0) (2,2)

(3,1) (3,3)

(4,0) (4,2) (4,4)

(5,1) (5,3) (5,5)

(6,0) (6,2) (6,4) (6,6)

(7,1) (7,3) (7,5) (7,7)

(8,0) (8,2) (8,4) (8,6) (8,8)

(9,1) (9,3) (9,5) (9,7) (9,9)

(10,0) (10,2) (10,4) (10,6) (10,8) (10,10)

(11,1) (11,3) (11,5) (11,7) (11,9) (11,11)

(12,0) (12,2) (12,4) (12,6) (12,8) (12,10) (12,12)

The values of  $(N, d)$  where Jordan blocks appear for  $\beta = 0$ .

Two conditions for Jordan blocks:

- $i_N^d$  is not an isomorphism;
- Raising and lowering operators of  $U_q(sl_2)$  commute with  $H_{XXZ}$  (if  $u^{4P} = 1$  only).

# Jordan cells

(1,1)

(2,0) (2,2)

(3,1) (3,3)

(4,0) (4,2) (4,4)

(5,1) (5,3) (5,5)

(6,0) (6,2) (6,4) (6,6)

(7,1) (7,3) (7,5) (7,7)

(8,0) (8,2) (8,4) (8,6) (8,8)

(9,1) (9,3) (9,5) (9,7) (9,9)

(10,0) (10,2) (10,4) (10,6) (10,8) (10,10)

(11,1) (11,3) (11,5) (11,7) (11,9) (11,11)

(12,0) (12,2) (12,4) (12,6) (12,8) (12,10) (12,12)

The values of  $(N, d)$  where Jordan blocks appear for  $\beta = 0$ .

Two conditions for Jordan blocks:

- $i_N^d$  is not an isomorphism;
- Raising and lowering operators of  $U_q(sl_2)$  commute with  $H_{XXZ}$  (if  $u^{4p} = 1$  only).

Thank you  
for your attention!