The Jordan structure of periodic loop models

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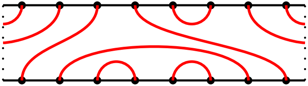
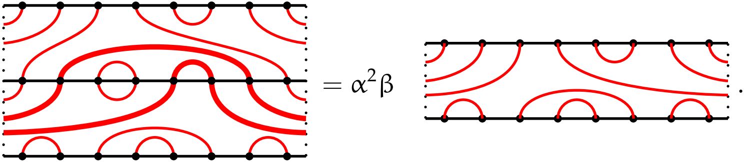
Joint work with Yvan Saint-Aubin

The periodic Temperley-Lieb algebra $TLP_N(\alpha, \beta)$

A connectivity is a set of non-intersecting curves connecting $2N$ nodes, N on the top and N on the bottom of a periodic strip.



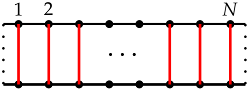
The product between connectivities is given by

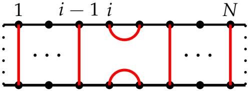


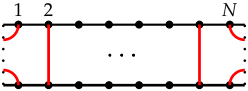
$TLP_N(\alpha, \beta)$ is the vector space generated by connectivities and endowed with this product.

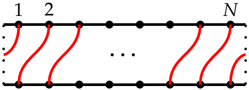
The algebra $TLP_N(\alpha, \beta)$

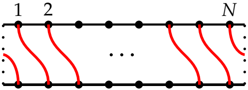
The Temperley-Lieb algebra can be generated by

$id =$ 

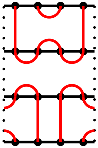
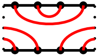
$e_i =$ 

$e_0 =$ 

$\Omega =$ 

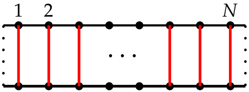
$\Omega^{-1} =$ 

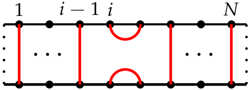
Any connectivity is obtained by a multiplication of some generators.

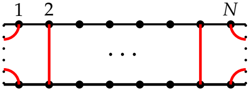
$e_0 e_1 e_3 e_2 =$  $=$ 

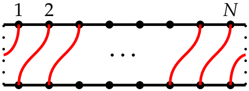
The algebra $TLP_N(\alpha, \beta)$

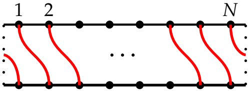
The Temperley-Lieb algebra can be generated by

$id =$ 

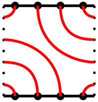
$e_i =$ 

$e_0 =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

Curves can wind around the cylinder indefinitely, so $TLP_N(\alpha, \beta)$ is infinite dimensional!

$e_0 e_1 e_2 e_3 e_0 =$ 

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

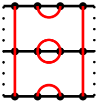
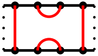
$$\begin{aligned} e_i^2 &= \beta e_i \\ e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i \quad (|i-j| > 1) \\ e_N &= e_0 \\ \Omega\Omega^{-1} &= \Omega^{-1}\Omega = id \end{aligned}$$

$$\begin{aligned} \Omega e_i \Omega^{-1} &= e_{i-1}, \\ e_{N-1}e_{N-2}...e_2 e_1 &= \Omega^2 e_1 \\ e_1 e_2 ... e_{N-2} e_{N-1} &= \Omega^{-2} e_{N-1} \\ E\Omega^{\pm 1} E &= \alpha E \\ (\text{where } E &= e_0 e_2 e_4 ... e_{N-2}) \end{aligned}$$

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$
\begin{aligned}
e_i^2 &= \beta e_i \\
e_i e_{i\pm1} e_i &= e_i \\
e_i e_j &= e_j e_i \quad (|i-j| > 1) \\
e_N &= e_0 \\
\Omega\Omega^{-1} &= \Omega^{-1}\Omega = id
\end{aligned}
\qquad
\begin{aligned}
\Omega e_i \Omega^{-1} &= e_{i-1}, \\
e_{N-1} e_{N-2} \ldots e_2 e_1 &= \Omega^2 e_1 \\
e_1 e_2 \ldots e_{N-2} e_{N-1} &= \Omega^{-2} e_{N-1} \\
E\Omega^{\pm1} E &= \alpha E \\
(\text{where } E &= e_0 e_2 e_4 \ldots e_{N-2})
\end{aligned}
$$

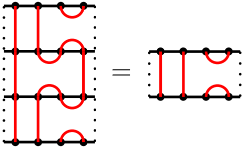
$(e_2)^2 =$  $= \beta$  $= \beta e_2$

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$
$$e_i e_{i\pm1} e_i = e_i$$
$$e_i e_j = e_j e_i \quad (|i-j| > 1)$$
$$e_N = e_0$$
$$\Omega\Omega^{-1} = \Omega^{-1}\Omega = id$$

$$\Omega e_i \Omega^{-1} = e_{i-1},$$
$$e_{N-1}e_{N-2}\dots e_2 e_1 = \Omega^2 e_1$$
$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$
$$E\Omega^{\pm1}E = \alpha E$$
$$(\text{where } E = e_0 e_2 e_4 \dots e_{N-2})$$

$e_3 e_2 e_3 =$  $= e_3$

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$
\begin{aligned}
e_i^2 &= \beta e_i \\
e_i e_{i\pm1} e_i &= e_i \\
e_i e_j &= e_j e_i \quad (|i-j| > 1) \\
e_N &= e_0 \\
\Omega\Omega^{-1} &= \Omega^{-1}\Omega = id
\end{aligned}
$$

$$
\begin{aligned}
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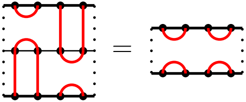
e_1e_2...e_{N-2}e_{N-1} &= \Omega^{-2} e_{N-1} \\
E\Omega^{\pm1}E &= \alpha E \\
(\text{where } E &= e_0e_2e_4...e_{N-2})
\end{aligned}
$$

$e_3e_1 =$  $= e_1e_3$

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

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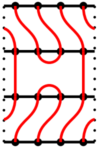
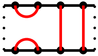
$$e_1e_2...e_{N-2}e_{N-1} = \Omega^{-2}e_{N-1}$$
$$E\Omega^{\pm1}E = \alpha E$$
$$(\text{where } E = e_0e_2e_4...e_{N-2})$$

$\Omega e_2 \Omega^{-1} =$  $=$  $= e_1$

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$
$$e_i e_{i\pm 1} e_i = e_i$$
$$e_i e_j = e_j e_i \quad (|i-j| > 1)$$
$$e_N = e_0$$
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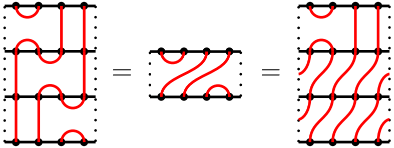
$$e_1 e_2 \dots e_{N-2} e_{N-1} = \Omega^{-2} e_{N-1}$$
$$E\Omega^{\pm 1} E = \alpha E$$
$$(\text{where } E = e_0 e_2 e_4 \dots e_{N-2})$$

$e_3 e_2 e_1 =$  $= \Omega^2 e_1$

The algebra $TLP_N(\alpha, \beta)$

The definition of the product can be done in terms of the generators:

$$e_i^2 = \beta e_i$$
$$e_i e_{i\pm1} e_i = e_i$$
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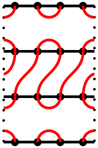
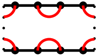
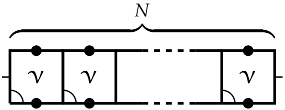
$E\Omega E =$  $= \alpha$  $= \alpha E$

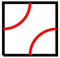
The transfer matrix and Hamiltonian

The transfer matrix and Hamiltonian are elements of $TLP_N(\alpha, \beta)$.

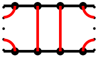
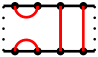
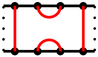
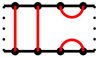
Transfer matrix:

$T_N(\lambda, \nu) =$ 

where 

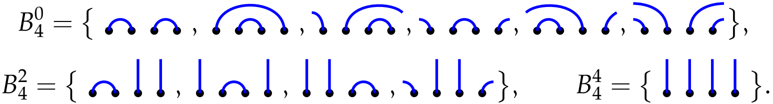
$= \sin(\lambda - \nu)$  $+ \sin\nu$ 

Hamiltonian:

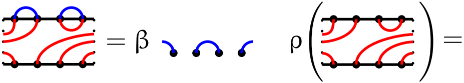
$\mathcal{H} = \sum_{i=0}^{N-1} e_i =$  $+$  $+$  $+$  .

Link states and the representation ρ

A representation of $TLP_N(\alpha, \beta)$ is obtained by defining the link states and the action of connectivities on link states:



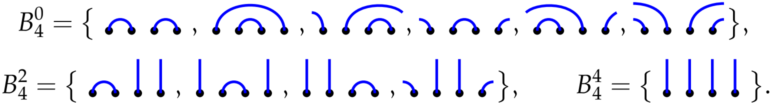
The action of $TLP_N(\alpha, \beta)$ elements on link states:



$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
\beta&1&0&0&\alpha&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&\beta&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&v^8&\beta v^6&v^8&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Link states and the representation ρ

A representation of $TLP_N(\alpha, \beta)$ is obtained by defining the link states and the action of connectivities on link states:



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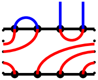
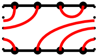


$= 0$

$\rho\Big($  $\Big) =$

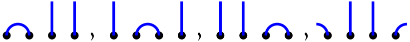
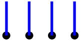
$$\left(\begin{array}{cccccc|cccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\beta & 1 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & v^8 & \beta v^6 & v^8 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

$d=0$, $d=2$, $d=4$

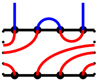
Link states and the representation ρ

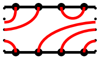
A representation of $TLP_N(\alpha, \beta)$ is obtained by defining the link states and the action of connectivities on link states:

$B_4^0 = \{$  $\}$,

$B_4^2 = \{$  $\}$, $B_4^4 = \{$  $\}$.

The action of $TLP_N(\alpha, \beta)$ elements on link states:

 $= v^8$ 

$\rho\Big($  $\Big) =$

$$\left(\begin{array}{cccccc|cccc|c}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
\beta&1&0&0&\alpha&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&\beta&1&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&v^8&\beta v^6&v^8&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)$$

$d=0$, $d=2$, $d=4$

v is the twist parameter.

The Hamiltonian

$\mathcal{H}$ in the link representation:

$$\rho(\mathcal{H}) = \left(\begin{array}{cccccc|cccc|c}
2\beta & 2 & \alpha & 0 & \alpha & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 2 & 2\beta & 2 & \alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \beta & v^2 & 0 & v^{-2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & \beta & v^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & \beta & v^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & v^{-2} & \beta & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

For specific values of α, β and v
$\mathcal{H}$ is non diagonalizable!

For example: $\beta = 0, \alpha = 2$,
a rank 2 Jordan cell
appears in the $d = 0$ sector.

$$S\rho(\mathcal{H})S^{-1} = \left(\begin{array}{cccccc|cccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Jordan cells for finite N $\quad\rightarrow\quad$ Logarithmic CFT in the scaling limit

The XXZ Hamiltonian

The generalized Hamiltonian is given by

$$H_{XXZ} = \frac{1}{2}\sum_{j=0}^{N-1}\Big(\frac{v^2+v^{-2}}{2}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) - \frac{v^2-v^{-2}}{2i}(\sigma_j^x\sigma_{j+1}^y - \sigma_j^y\sigma_{j+1}^x) + \frac{u^2+u^{-2}}{2}(\sigma_j^z\sigma_{j+1}^z - id)\Big)$$

- $\sigma_j^a = \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{j-1} \otimes \sigma^a \otimes \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{N-j}$.
- $\sigma_N^a \equiv \sigma_0^a$.
- It acts on $(\mathbb{C}^2)^{\otimes N}$. For $N = 4$: $|\downarrow\downarrow\downarrow\downarrow\rangle, |\downarrow\downarrow\downarrow\uparrow\rangle, |\downarrow\downarrow\uparrow\downarrow\rangle, \ldots, |\uparrow\uparrow\uparrow\uparrow\rangle$.
- The usual case is just $v^2 = 1$ and $\Delta = \frac{u^2+u^{-2}}{2}$.
- H_{XXZ} is diagonalizable for $u = e^{i\phi}, v = e^{i\gamma}$ and $\gamma, \phi \in \mathbb{R}$.

The XXZ representation of TLP_N

It can be rewritten as

$$H_{XXZ} = \sum_{j=0}^{N-1} \bar{e}_j, \quad \text{with}$$

$$\bar{e}_j = \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u^2 & v^2 & 0 \\ 0 & v^{-2} & u^{-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{id_2 \otimes id_2 \otimes \cdots \otimes id_2}_{N-j-1}$$

- H_{XXZ} commutes with $S^z = \frac{1}{2} \sum_{j=0}^{N-1} \sigma_j^z$.
- Let t be the operator that translates all spins one position to the left. The matrices $\bar{e}_j$, along with $\bar{\Omega} = v^{2S^z} t$ satisfy all the relations of $TLP_N(\alpha, \beta)$, for $\beta = u^2 + u^{-2}$, and $\alpha = v^N + v^{-N}$ for N even.

The XXZ Hamiltonian: an example for $N = 4$

In the spin basis $\{\ldots, |\uparrow\uparrow\downarrow\downarrow\rangle, |\uparrow\downarrow\uparrow\downarrow\rangle, |\uparrow\downarrow\downarrow\uparrow\rangle, |\downarrow\uparrow\uparrow\downarrow\rangle, |\downarrow\uparrow\downarrow\uparrow\rangle, |\downarrow\downarrow\uparrow\uparrow\rangle,$
$|\uparrow\uparrow\uparrow\downarrow\rangle, |\uparrow\uparrow\downarrow\uparrow\rangle, |\uparrow\downarrow\uparrow\uparrow\rangle, |\downarrow\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\uparrow\uparrow\rangle\},$

$$\left(\begin{array}{ccccccc|cccc|c}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ldots & u^2+u^{-2} & v^2 & 0 & 0 & v^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & v^{-2} & 2(u^2+u^{-2}) & v^2 & v^2 & v^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & 0 & v^{-2} & u^2+u^{-2} & 0 & v^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & 0 & v^{-2} & 0 & u^2+u^{-2} & v^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ldots & v^2 & 0 & v^{-2} & v^{-2} & 2(u^2+u^{-2}) & v^2 & 0 & 0 & 0 & 0 & 0 \\
\ldots & 0 & v^2 & 0 & 0 & v^{-2} & u^2+u^{-2} & 0 & 0 & 0 & 0 & 0 \\ \cline{2-12}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & u^2+u^{-2} & v^2 & 0 & v^{-2} & 0 \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & u^2+u^{-2} & v^2 & 0 & 0 \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^{-2} & u^2+u^{-2} & v^2 & 0 \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & v^2 & 0 & v^{-2} & u^2+u^{-2} & 0 \\ \cline{2-12}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

The XXZ Hamiltonian: an example for $N = 4$

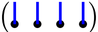
Setting $u = e^{i\pi/4}(\beta = 0)$ and $v = 1(\alpha = 2)$, it can be diagonalized:

$$\bar{S} H_{XXZ} \bar{S}^{-1} = \left(\begin{array}{cccccc|cccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

This is the same as $S\rho(H)S^{-1}$ except for the Jordan block!

The map i_N^d

i_N^d: link states with n bubbles $\to$ spin states with n down arrows.

$i_4^4\Big($  $\Big) = |\uparrow\uparrow\uparrow\uparrow\rangle,$

$i_4^2\Big($  $\Big) = (uv\sigma_3^- + u^{-1}v^{-1}\sigma_2^-)|\uparrow\uparrow\uparrow\uparrow\rangle$

$= uv\,|\uparrow\uparrow\downarrow\uparrow\rangle + (uv)^{-1}\,|\uparrow\downarrow\uparrow\uparrow\rangle,$

$i_4^0\Big($  $\Big) = (uv\sigma_2^- + u^{-1}v^{-1}\sigma_1^-)(uv\sigma_4^- + u^{-1}v^{-1}\sigma_3^-)|\uparrow\uparrow\uparrow\uparrow\rangle$

$= (uv)^2\,|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + (uv)^{-2}\,|\downarrow\uparrow\downarrow\uparrow\rangle,$

$i_4^0\Big($  $\Big) = (uv^3\sigma_4^- + u^{-1}v^{-3}\sigma_1^-)(uv\sigma_3^- + u^{-1}v^{-1}\sigma_2^-)|\uparrow\uparrow\uparrow\uparrow\rangle$

$= u^2v^4\,|\uparrow\uparrow\downarrow\downarrow\rangle + v^2\,|\uparrow\downarrow\uparrow\downarrow\rangle + v^{-2}\,|\downarrow\uparrow\downarrow\uparrow\rangle + u^{-2}v^{-4}\,|\downarrow\downarrow\uparrow\uparrow\rangle.$

The transformation i_N^d

Proposition 1

i_N^d is a homomorphism if $\beta = u^2 + u^{-2}$ and $\alpha = v^N + v^{-N}$:
$i_N^d(e_i v) = \bar{e}_i i_N^d(v)$ and $i_N^d(\Omega v) = \bar{\Omega} i_N^d(v)$ for any link state v.

Proposition 2

i_N^d is an isomorphism except if u and v are such that

$$\prod_{k=1}^{(N-d)/2} \left((iu)^{4k+2d} v^{2N} - 1 \right) = 0$$

- $\rho(\mathcal{H})$ and H_{XXZ} have the same spectrum if $\beta = u^2 + u^{-2}$ and $\alpha = v^N + v^{-N}$.
- When i_N^d is an isomorphism, $\rho(\mathcal{H})$ is diagonalizable.

Jordan cells

(1,1)

(2,0) (2,2)

(3,1) (3,3)

(4,0) (4,2) (4,4)

(5,1) (5,3) (5,5)

(6,0) (6,2) (6,4) (6,6)

(7,1) (7,3) (7,5) (7,7)

(8,0) (8,2) (8,4) (8,6) (8,8)

(9,1) (9,3) (9,5) (9,7) (9,9)

(10,0) (10,2) (10,4) (10,6) (10,8) (10,10)

(11,1) (11,3) (11,5) (11,7) (11,9) (11,11)

(12,0) (12,2) (12,4) (12,6) (12,8) (12,10) (12,12)

The values of (N, d) where Jordan blocks appear for $\beta = 0$.

Two conditions for Jordan blocks:

- i_N^d is not an isomorphism;
- Raising and lowering operators of $U_q(sl_2)$ commute with H_{XXZ} (if $u^{4P} = 1$ only).

Jordan cells

(1,1)

(2,0) (2,2)

(3,1) (3,3)

$\boxed{(4,0)}$ (4,2) (4,4)

$\boxed{(5,1)}$ (5,3) (5,5)

$\boxed{(6,0)}$ $\boxed{(6,2)}$ (6,4) (6,6)

$\boxed{(7,1)}$ $\boxed{(7,3)}$ (7,5) (7,7)

$\boxed{\boxed{(8,0)}}$ $\boxed{(8,2)}$ $\boxed{(8,4)}$ (8,6) (8,8)

$\boxed{\boxed{(9,1)}}$ $\boxed{(9,3)}$ $\boxed{(9,5)}$ (9,7) (9,9)

$\boxed{\boxed{(10,0)}}$ $\boxed{\boxed{(10,2)}}$ $\boxed{(10,4)}$ $\boxed{(10,6)}$ (10,8) (10,10)

$\boxed{\boxed{(11,1)}}$ $\boxed{\boxed{(11,3)}}$ $\boxed{(11,5)}$ $\boxed{(11,7)}$ (11,9) (11,11)

$\boxed{\boxed{\boxed{(12,0)}}}$ $\boxed{\boxed{(12,2)}}$ $\boxed{\boxed{(12,4)}}$ $\boxed{(12,6)}$ $\boxed{(12,8)}$ (12,10) (12,12)

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Thank you
for your attention!