Feigin–Frenkel center and Yangian characters

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Invariants in vacuum modules

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Define the invariant bilinear form on a simple Lie algebra $\mathfrak{g}$,

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For the classical types, $\langle X, Y\rangle = \mathrm{const} \cdot \operatorname{tr} X Y$,

$$h^{\vee} = \begin{cases} n & \text{for} \quad \mathfrak{g} = \mathfrak{sl}_n, \qquad \mathrm{const} = 1 \\ N-2 & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N, \qquad \mathrm{const} = \frac{1}{2} \\ n+1 & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}, \qquad \mathrm{const} = 1. \end{cases}$$

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The vacuum module at the critical level $V(\mathfrak{g})$ over $\widehat{\mathfrak{g}}$ is the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K + h^\vee$.

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Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector.

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- The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is invariant with respect to the translation operator T defined as the derivation $T = -\partial_t$.

Theorem (Feigin–Frenkel, 1992).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}])$,

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We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

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- Produce Segal–Sugawara vectors $S_1, \dots, S_n$ explicitly.
- Show that all elements $T^k S_l$ with $l = 1, \dots, n$ and $k \geqslant 0$ are algebraically independent.
- Show that they generate $\mathfrak{z}(\widehat{\mathfrak{g}})$ by taking the classical limit.

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$$P_{(r)} = T^r P\big(X_1[-1], \dots, X_d[-1]\big), \qquad r \geqslant 0,$$

is a $\mathfrak{g}[t]$-invariant in the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Theorem (Beilinson–Drinfeld, 1997). If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then the elements $P_{1,(r)}, \dots, P_{n,(r)}$ with $r \geqslant 0$ are algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

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Example. Let $\mathfrak{g} = \mathfrak{sl}_2$ with the basis e, f, h. Then $P = h^2 + 4fe$ is the generator of $\mathrm{S}(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$. The algebra of $\mathfrak{sl}_2[t]$-invariants in $\mathrm{S}\big(t^{-1}\mathfrak{sl}_2[t^{-1}]\big)$ is generated by the elements $P_{(r)}$ with $r \geqslant 0$,

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$$P_{(0)} = h[-1]^2 + 4f[-1]\,e[-1],$$

$$P_{(1)} = 2h[-1]\,h[-2] + 4f[-2]\,e[-1] + 4f[-1]\,e[-2], \quad \textit{etc.}$$

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and let $H^{(m)}$ and $A^{(m)}$ denote the symmetrizer and anti-symmetrizer in $\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}.$

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$$\operatorname{tr} A^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big)$$

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[Chervov–Talalaev, 2006, Chervov–M., 2009].

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Brauer algebra $\mathcal{B}_m(\omega)$

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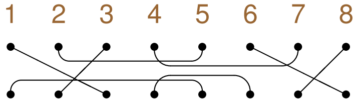

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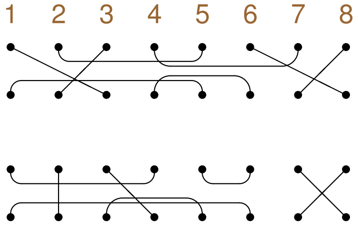

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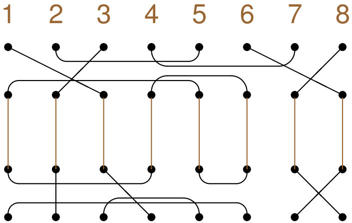

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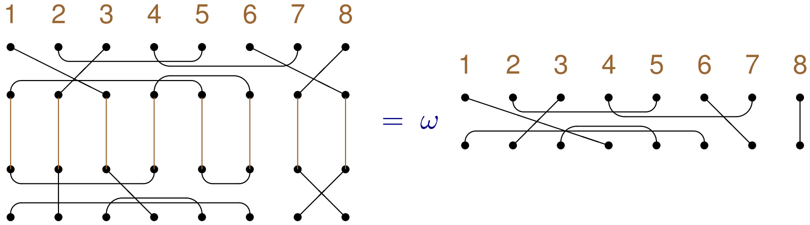

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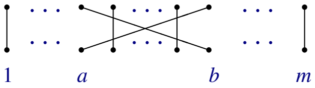

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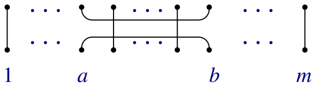

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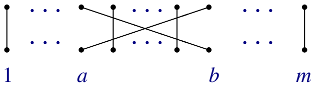

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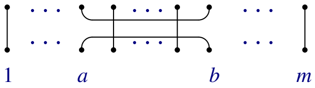

The symmetrizer in the Brauer algebra $\mathcal{B}_m(\omega)$

is the idempotent $s^{(m)}$ such that

$$s_{ab}\, s^{(m)} = s^{(m)}\, s_{ab} = s^{(m)} \qquad \text{and} \qquad \epsilon_{ab}\, s^{(m)} = s^{(m)}\, \epsilon_{ab} = 0.$$

Action in tensors

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In the case $\mathfrak{g} = \mathfrak{o}_N$ set $\omega = N$. The generators of $\mathcal{B}_m(N)$ act in the tensor space

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by the rule

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In the case $\mathfrak{g} = \mathfrak{sp}_N$ with $N = 2n$ set $\omega = -N$. The generators of $\mathcal{B}_m(-N)$ act in the tensor space $(\mathbb{C}^N)^{\otimes m}$ by

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In both cases denote by $S^{(m)}$ the image of the symmetrizer $s^{(m)}$ under the action in tensors,

$$S^{(m)} \in \underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m}.$$

Explicitly,

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \right),$$

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$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 - \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{n - b + a + 1} \right).$$

Set

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2}, \qquad \omega = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N \\ -2n & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

Theorem. All coefficients of the polynomial in $\tau = -d/dt$

$$\gamma_m(\omega)\,\mathrm{tr}\,S^{(m)}\big(\tau + F[-1]_1\big)\ldots\big(\tau + F[-1]_m\big)$$

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Moreover, in the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Pfaffian

$$\mathrm{Pf}\, F[-1] = \frac{1}{2^n n!} \sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma\cdot F_{\sigma(1)\,\sigma(2)'}[-1]\dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1].$$

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where $V_1, \dots, V_n$ are the screening operators in

$\mathbb{C}\left[\mu_1[-r], \dots, \mu_n[-r] \;\middle|\; r \geqslant 1\right]$.

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where $\mathcal{W}({}^L\mathfrak{g})$ is the classical $\mathcal{W}$-algebra associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ [Feigin and Frenkel, 1992].

Given ordered variables $x_1, \ldots, x_N$, set

$$h_m(x_1, \ldots, x_N) = \sum_{i_1 \leqslant \cdots \leqslant i_m} x_{i_1} \ldots x_{i_m},$$

$$e_m(x_1, \ldots, x_N) = \sum_{i_1 > \cdots > i_m} x_{i_1} \ldots x_{i_m}.$$

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For $\mathfrak{g} = \mathfrak{gl}_N$, under the Harish-Chandra isomorphism,

$$\operatorname{tr} A^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big)$$

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The Harish-Chandra image of the polynomial

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$$\tfrac{1}{2}\,h_m\big(\tau+\mu_1[-1],\dots,\tau+\mu_{n-1}[-1],\tau-\mu_n[-1],\dots\tau-\mu_1[-1]\big)$$

$$+\tfrac{1}{2}\,h_m\big(\tau+\mu_1[-1],\dots,\tau+\mu_n[-1],\tau-\mu_{n-1}[-1],\dots\tau-\mu_1[-1]\big),$$

for the Lie algebra $\mathfrak{g}=\mathfrak{o}_N$ with $N=2n$.

The Harish-Chandra image of the polynomial

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with $1 \leqslant m \leqslant 2n$ equals:

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for the Lie algebra $\mathfrak{g}=\mathfrak{sp}_{2n}$.

In the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Harish-Chandra image of the Pfaffian

$$\operatorname{Pf} F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

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is found by

$$\mathrm{Pf}\, F[-1] \mapsto \big(\mu_1[-1] - \tau\big) \dots \big(\mu_n[-1] - \tau\big)\, 1.$$

Corollary. The elements $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ form a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$.

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Calculation of Harish-Chandra images

Bethe subalgebra
[transfer matrices]

Yangian characters
[Grothendieck ring]

$$\begin{array}{ccc} \mathcal{B}(\mathfrak{g}) & \xrightarrow[\text{isomorphism}]{\text{Harish-Chandra}} & \operatorname{char} \mathrm{Y}(\mathfrak{g}) \\ \Big\downarrow {\scriptstyle \text{classical limit}} & & \Big\downarrow {\scriptstyle \text{classical limit}} \\ \mathfrak{z}(\widehat{\mathfrak{g}}) & \xrightarrow[\text{isomorphism}]{\text{Harish-Chandra}} & \mathcal{W}({}^L\mathfrak{g}) \end{array}$$

Feigin–Frenkel center
[Segal–Sugawara vectors]

classical $\mathcal{W}$-algebra

The space of (skew-)symmetric harmonic tensors

$$S^{(m)}(\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m})$$

is an irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{o}_N)$ or $\mathrm{Y}(\mathfrak{sp}_N)$.

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For $\mathrm{Y}(\mathfrak{o}_N)$ this is one of the Kirillov–Reshetikhin modules;

for $\mathrm{Y}(\mathfrak{sp}_N)$ this is one of the fundamental modules.

For $\mathrm{Y}(\mathfrak{o}_N)$ their q-characters are given by

$$\sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_m \leqslant N} \lambda_{i_1}(u)\, \lambda_{i_2}(u+1) \dots \lambda_{i_m}(u+m-1),$$

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with different conditions for B_n and D_n:

- $\mathfrak{o}_{2n+1}$: index $n+1$ occurs at most once;
- $\mathfrak{o}_{2n}$: indices n and $n+1$ do not occur simultaneously.

For $\mathrm{Y}(\mathfrak{sp}_{2n})$ the q-character is given by

$$\sum_{1\leqslant i_1<\dots<i_m\leqslant 2n} \lambda_{i_1}(u)\,\lambda_{i_2}(u-1)\dots\lambda_{i_m}(u-m+1),$$

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Use an equivalent form of the q-character due to [Kuniba–Okado–Suzuki–Yamada, 2002].