Non–Hermitian BCS pairing Hamiltonian and Generalised Exclusion Statistics

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Lorne, December 2012

Overview

J. Links, A. Moghaddam and Y.Z. Zhang ; " *Deconfined quantum criticality and generalized exclusion statistics in a non-Hermitian BCS model* " ; **J. Phys. A: Math. Theor. 45 (2012) 462002.**

- Introduction
  1. The story of Hermiticity
  2. Exactly Solvable Quantum Models (ESQM)
  3. BCS Model
- What have we done?
- Conclusion

The story of Hermiticity

- Hermiticity mathematically guarantees an operator's eigenvalues to be real.
- Since the 1950s, *non-Hermitian Hamiltonians* with real spectrum have been identified and applied in various contexts.

Year	Author(s)	Reference	Explanation
1959	T. T. Wu	*"Ground State of a Bose System of Hard Spheres"*, Phys. Rev. **115**, 1390.	A non-Hermitian Hamiltonians for the ground state of Bose system of hard spheres
1992	T. Hollowood	*"Solitons in affine Toda field theory"*, Nucl.Phys. **B384**, 523.	a non-Hermitian Hamiltonians for complex Toda lattice
1998	C.M.Bender S.Boettcher	*"Real spectra in non-Hermitian Hamiltonians having PT symmetries"*, Phys.Rev.Lett. **80**, 5243.	Describing classical and quantum properties of non-Hermitian Hamiltonians PT–symmetric Hamiltonian.
2007	C.Korff R.A.Weston	*"PT Symmetry on the Lattice: The Quantum Group invariant XXZ spin-chain"*, J.Phys. **A40**, 8845.	Connecting integrable lattice systems and non–Hermitian Hamiltonians.
2012	C. M. Bender V. Branchina E. Messina	*"Ordinary versus PT–symmetric ϕ^3 quantum field theory"*, `arXiv:1201.1244v1 [hep-th]`.	Discussing the properties of an analogue of the PT–symmetric quantum-mechanics described by the Hamiltonian $p^2 + ix^3$ in quantum field theory.

Exactly Solvable Quantum Models

An *Exactly Solvable Quantum Model* means a model Hamiltonian whose eigenvalues and eigenstates are exactly determent by a method known as *Bethe ansatz*.

- The 1D Heisenberg spin chain model was solved by H. A. Bethe in 1931.
- The 1D Bose gas model was solved by E. Lieb and W. Liniger in 1963.
- A reduced BCS pairing Hamiltonian was solved by R. W. Richardson in 1963.
- The 1D Hubbard model was solved by E. Lieb and F. Wu in 1968.

The Theory of Superconductivity

Bardeen, Cooper and Schrieffer (BCS) theory was presented in 1957. It became one of the most successful theories in the area of Superconductivity:

$$H_{BCS} = \sum_{j=1}^{L} \epsilon_j n_j - \sum_{j,k=1}^{L} G_{jk} c_{k+}^{\dagger} c_{k-}^{\dagger} c_{j-} c_{j+}.$$

where:

j: varies from 1 to L, labels a shell of doubly degenerate particle energy levels with energy ϵ_j.

$c_{j\pm}$,$c_{j\pm}^{\dagger}$: are the annihilation and creation operators for the fermions at level j. $\pm$ refer to time–reversed pairs.

n_j: equal to $c_{j+}^{\dagger} c_{j+} + c_{j-}^{\dagger} c_{j-}$ is the fermion number operator for level j.

G_{jk}: coupling variables.

Russian Doll BCS Model

Russian dolls are a set of wooden dolls which can be pulled apart to reveal another figurine of the same sort inside.

The following model (A. LeClair et al., 2004) has been named the *Russian doll BCS* model, since its renormalisation group flow is one which displays a cyclic nature rather than flowing to a fixed point

$$H_{RD} = \sum_{j=1}^{L} \epsilon_j n_j - G \sum_{j<k}^{L} (e^{i\alpha} c_{k+}^{\dagger} c_{k-}^{\dagger} c_{j-} c_{j+} + e^{-i\alpha} c_{j+}^{\dagger} c_{j-}^{\dagger} c_{k-} c_{k+}).$$

Its integrability was shown by C. Dunning and J. Links, 2004.

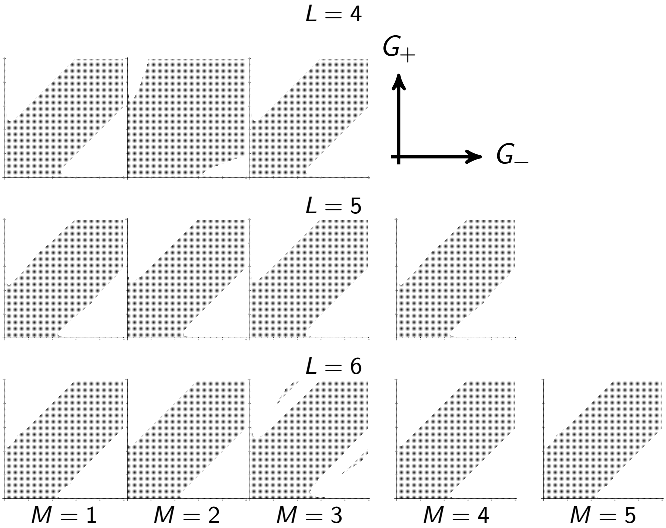
A Non–Hermitian Variant of BCS pairing Hamiltonian

In connection with the BCS Hamiltonian, the coupling variables can accept different values:

$$G_{jk} = \begin{cases} G_+ & j < k \\ \dfrac{G_+ + G_-}{2} & j = k \\ G_- & j > k \end{cases}$$

We will choose the $\epsilon_j = \left(j - \frac{L+1}{2}\right)\delta$ to be uniformly and symmetrically distributed around zero where $\delta > 0$ provides the level spacing. With respect to G_+ and G_- the possible ESQMs are:

G_+ & G_-	Model	Self–adjoint Hamiltonian	Real spectrum
Both real & equal	Richardson	✓	✓
Complex conjugate pair	Russian Doll	✓	✓
Both real	Ours	✗	?



Boundary lines: $G_+ - G_- = \pm 2\delta$

Solubility & BA of the new model

The exact solution for the model was obtained by Quantum Inverse Scattering Method and algebraic Bethe ansatz and adapted from RD-model integrability. To describe the exact solution, we consider:

$$G_+ = \frac{2\eta e^{\alpha}}{e^{\alpha} - e^{-\alpha}} \quad ; \quad G_- = \frac{2\eta e^{-\alpha}}{e^{\alpha} - e^{-\alpha}}$$

It turns out that the exact solution of energy spectrum is:

$$E = 2\sum_{j=1}^{M} v_j$$

where the v_j are Bethe ansatz solutions:

$$e^{2\alpha} P(v_k + \eta) \prod_{j \neq k}^{M} (v_k - v_j - \eta) = P(v_k) \prod_{j \neq k}^{M} (v_k - v_j + \eta) \, ; \; k = 1, \ldots, M$$

where: $P(u) = \prod_{j=1}^{L} (u - \epsilon_j - \eta/2)$.

A Special Case

$G_+ - G_- = \pm 2\delta \Rightarrow \eta = \pm\delta$, so, $\epsilon_j = \eta(j - \frac{L+1}{2})$, the Bethe ansatz equations become:

$$\prod_{l=1}^{L-1}\left(v_k - \eta\left(l - \frac{L}{2}\right)\right)\left(e^{\alpha}\left(v_k + \frac{\eta L}{2}\right)\prod_{j\neq k}^{M}(v_k - v_j - \eta) - e^{-\alpha}\left(v_k - \frac{\eta L}{2}\right)\prod_{j\neq k}^{M}(v_k - v_j + \eta)\right) = 0$$

In this form it is clear that we obtain a solution set by choosing $v_k \in \mathcal{S}$, $k = 1, ..., M$, for

$$\mathcal{S} = \{\eta(j - L/2) : j = 1, ..., L-1\}.$$

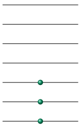
L	M	E	v_k
4	1	-2η	$-\eta$
		0	0
		2η	η
		$-\frac{4\left(1+e^{2\alpha}\right)\eta}{-1+e^{2\alpha}}$	$-2\eta\,\frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}$
	2	$-\frac{2\left(1+e^{2\alpha}\right)\eta}{-1+e^{2\alpha}}$	$(-\eta,\ -\frac{2\eta}{-1+e^{2\alpha}})$ $(\eta,\ -\frac{2e^{2\alpha}\eta}{-1+e^{2\alpha}})$
		$-\frac{6\left(1+e^{2\alpha}\right)\eta}{-1+e^{2\alpha}}$	$\left(\frac{\left(-3-3e^{2\alpha}+\sqrt{1-14e^{2\alpha}+e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)},\ \frac{\left(-3-3e^{2\alpha}-\sqrt{1-14e^{2\alpha}+e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)}\right)$
		0	$(\eta,-\eta)$
		$\frac{-\eta-e^{2\alpha}\eta-\sqrt{9\eta^2-14e^{2\alpha}\eta^2+9e^{4\alpha}\eta^2}}{-1+e^{2\alpha}}$	$\left(0,\ \frac{\eta+e^{2\alpha}\eta+\sqrt{\left(9-14e^{2\alpha}+9e^{4\alpha}\right)\eta^2}}{2-2e^{2\alpha}}\right)$
		$\frac{-\eta-e^{2\alpha}\eta+\sqrt{9\eta^2-14e^{2\alpha}\eta^2+9e^{4\alpha}\eta^2}}{-1+e^{2\alpha}}$	$\left(0,\ \frac{\eta+e^{2\alpha}\eta-\sqrt{\left(9-14e^{2\alpha}+9e^{4\alpha}\right)\eta^2}}{2-2e^{2\alpha}}\right)$

L	M	E	v_k
5	2	$-\frac{2\left(2+e^{2\alpha}\right)\eta}{-1+e^{2\alpha}}$	$\left(\frac{3\eta}{2}, \frac{\eta+5e^{2\alpha}\eta}{2-2e^{2\alpha}}\right)$
		-2η	$\left(\frac{\eta}{2}, -\frac{3\eta}{2}\right)$
		$-\frac{2\left(1+2e^{2\alpha}\right)\eta}{-1+e^{2\alpha}}$	$\left(-\frac{3\eta}{2}, -\frac{\left(5+e^{2\alpha}\right)\eta}{2\left(-1+e^{2\alpha}\right)}\right)$
		2η	$\left(\frac{3\eta}{2}, -\frac{\eta}{2}\right)$
		$-\frac{8\left(1+e^{2\alpha}\right)\eta}{-1+e^{2\alpha}}$	$\left(-\frac{\left(4+4e^{2\alpha}+\sqrt{1-18e^{2\alpha}+e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)}, \frac{\left(-4-4e^{2\alpha}+\sqrt{1-18e^{2\alpha}+e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)}\right)$
		0	$\left(\frac{3\eta}{2}, -\frac{3\eta}{2}\right)$ and $\left(\frac{\eta}{2}, -\frac{\eta}{2}\right)$
		$\frac{\left(-3e^{2\alpha}+\sqrt{16-16e^{2\alpha}+9e^{4\alpha}}\right)\eta}{-1+e^{2\alpha}}$	$\left(-\frac{\eta}{2}, \frac{\left(-1-2e^{2\alpha}+\sqrt{16-16e^{2\alpha}+9e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)}\right)$
		$-\frac{\left(3e^{2\alpha}+\sqrt{16-16e^{2\alpha}+9e^{4\alpha}}\right)\eta}{-1+e^{2\alpha}}$	$\left(-\frac{\eta}{2}, \frac{\eta+2e^{2\alpha}\eta+\sqrt{16-16e^{2\alpha}+9e^{4\alpha}}\eta}{2-2e^{2\alpha}}\right)$
		$\frac{\left(-3+\sqrt{9+16e^{2\alpha}\left(-1+e^{2\alpha}\right)}\right)\eta}{-1+e^{2\alpha}}$	$\left(\frac{\eta}{2}, -\frac{\left(2+e^{2\alpha}-\sqrt{9-16e^{2\alpha}+16e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)}\right)$
		$-\frac{\left(3+\sqrt{9+16e^{2\alpha}\left(-1+e^{2\alpha}\right)}\right)\eta}{-1+e^{2\alpha}}$	$\left(\frac{\eta}{2}, -\frac{\left(2+e^{2\alpha}+\sqrt{9-16e^{2\alpha}+16e^{4\alpha}}\right)\eta}{2\left(-1+e^{2\alpha}\right)}\right)$

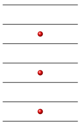
Bethe roots & GES

Based on our investigations, we identified a key property for the distribution and number of states for α independent roots.

Fermionic system



Our model



The number of ways that M Fermions occupy L levels : $\frac{L!}{(L-M)!M!}$

The number of ways that M quasi–particles occupy L levels : $\frac{(L-M)!}{(L-2M)!M!}$

The Fermionic system is equivalent to the first term of the Hamiltonian (i.e. the coupling variables G_+ and G_- are zero). By adding the second term with $G_+ - G_- = \pm 2\delta$ the energy levels vary in such a way that the particles reside in between the previous energy gaps and cannot be placed into adjacent levels.

Conclusion

1. It turns out that this many–particle system with non–Hermitian Hamiltonian yields a real spectrum for some regions in parameter space. This fact supports the proposition that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition.
2. This model possesses some remarkable properties which are absent in the case of the usual BCS model.
3. It is found that the α-independent spectrum of this model can be associated to exotic quasi-particles obeying generalised exclusion statistics, in the sense proposed by Haldane in 1991.

THANK YOU!