Elliptic parametrization of the Zamolodchikov model

Vladimir Mangazeev

Australian National University

ANZAMP Meeting, Lorne, 3 December 2012

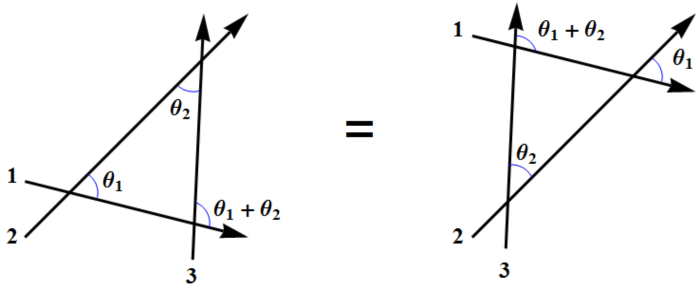
Collaboration:
V.Bazhanov, Y. Okada (ANU, Canberra) and S.Sergeev (UC, Canberra)

The tetrahedron equation is a 3D generalzation of the YBE equation.

- 3D Zamolodchikov model of straight strings (1981,1982)
    - A “static limit” solution of the TE (1981)
    - A “full” solution of the TE (1982)
- Baxter's results
    - Proof of the TE for the Zamolodchikov model (1983)
    - Criticality of the Zamolodchikov model (Baxter, Forrester, 1985)
    - Formulation as the 3D IRC statistical model, partition function (1986)
    - Hamiltonian limit (Baxter, Quispel, 1989)
- ZBB IRC N-state model (Bazhanov, Baxter (1992)
- First vertex results:
    - Korepanov's results (1993): tetrahedral Zamolodchikov algebra and its realization, the “static” vertex solution of the TE
    - Hietarinta's “planar” vertex solution of the TE
- A full vertex solution of the TE (Sergeev, M, Stroganov, 1995)
- Vertex-IRC duality of the SMS model and Zamolodchikov model (BMSS, 1996)

Vertex versions of the YBE and TE

The Yang-Baxter equation with a “difference” property

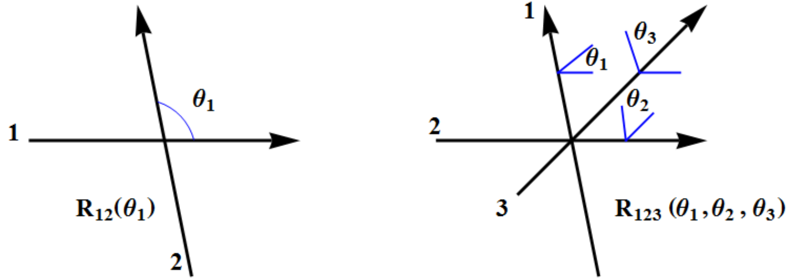


$$R_{12}(\theta_1)R_{13}(\theta_1+\theta_3)R_{23}(\theta_3) = R_{23}(\theta_3)R_{13}(\theta_1+\theta_3)R_{12}(\theta_1)$$

where for our case $R_{12}(\theta)$ is a linear operator acting in $\mathbb{C}^2 \times \mathbb{C}^2$.

Vertex versions of the YBE and TE

The most natural way to generalize to 3D is to replace R_{12} acting in $\mathbb{C}^2 \times \mathbb{C}^2$ by R_{123} acting in $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$



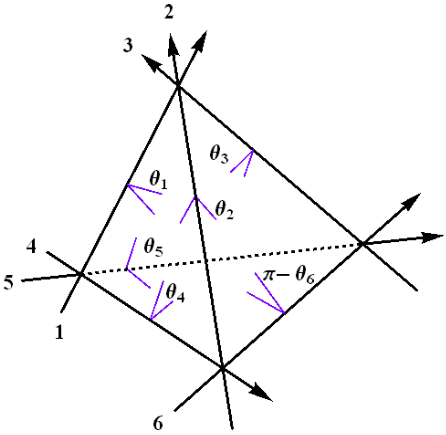
Vertex versions of the YBE and TE

$R \longrightarrow R(\theta_1, \theta_2, \theta_3)$

$R' \longrightarrow R(\theta_1, \theta_4, \theta_5)$

$R'' \longrightarrow R(\pi - \theta_2, \theta_4, \theta_6)$

$R''' \longrightarrow R(\theta_3, \pi - \theta_5, \theta_6)$



$+$A quadrilateral constraint

$$R_{123} R'_{145} R''_{246} R'''_{356} = R'''_{356} R''_{246} R'_{145} R_{123}$$

$$
\begin{array}{rrrrl}
R^{0,0,0}_{0,0,0} = & R^{0,1,1}_{0,1,1} = & R^{1,0,1}_{1,0,1} = & R^{1,1,0}_{1,1,0} = & 1 \\
R^{1,1,1}_{1,1,1} = & R^{1,0,0}_{1,0,0} = & R^{0,1,0}_{0,1,0} = & R^{0,0,1}_{0,0,1} = & t_0 t_1 t_2 t_3 \\
R^{0,1,0}_{0,0,1} = & R^{0,0,1}_{0,1,0} = & -R^{1,0,0}_{1,1,1} = & -R^{1,1,1}_{1,0,0} = & t_2 t_3 \\
R^{1,0,1}_{1,1,0} = & R^{1,1,0}_{1,0,1} = & -R^{0,1,1}_{0,0,0} = & -R^{0,0,0}_{0,1,1} = & t_0 t_1 \\
R^{1,1,1}_{0,1,0} = & R^{1,0,0}_{0,0,1} = & -R^{0,0,1}_{1,0,0} = & -R^{0,1,0}_{1,1,1} = & -\mathrm{i} t_1 t_3 \\
R^{0,0,0}_{1,0,1} = & R^{0,1,1}_{1,1,0} = & -R^{1,1,0}_{0,1,1} = & -R^{1,0,1}_{0,0,0} = & \mathrm{i} t_0 t_2 \\
R^{0,0,1}_{1,1,1} = & R^{1,1,1}_{0,0,1} = & R^{1,0,0}_{0,1,0} = & R^{0,1,0}_{1,0,0} = & t_1 t_2 \\
R^{1,1,0}_{0,0,0} = & R^{0,0,0}_{1,1,0} = & R^{0,1,1}_{1,0,1} = & R^{1,0,1}_{0,1,1} = & t_0 t_3
\end{array}
$$

where

$$\alpha_0 = \frac{\theta_1 + \theta_2 + \theta_3 - \pi}{2}\,, \quad \alpha_i = \theta_i - \alpha_0$$

and

$$t_i = \sqrt{\tan \frac{\alpha_i}{2}}\,.$$

Degenerate cases

- Korepanov's Static limit
  $S_{123}(\theta_1,\theta_2,\theta_3) = R(\theta_1,\theta_2,\theta_3)$ provided $\theta_1+\theta_2+\theta_3=\pi$. One can consistently satisfy this condition for all 4 weights leaving 4 independent parameters. Each weight depends on 2 angles.
- Hietarinta's Planar limit
  4 vertices of the tetrahedron belong to the same plane. Each vertex depends on 2 angles, there are 4 parameters in total.
- A new "infinite prizm" limit

$$\theta_1+\theta_2+\theta_3=\pi, \quad \theta_4,\theta_5,\theta_6 \quad \text{are arbitrary}$$

  Quadrilateral constraint is still nontirival and "kills" another parameter, say, θ_2. Independent parameters are $\theta_4,\theta_5,\theta_6$ and the angle between the vertical direction and the "bottom" (elliptic modulus).

Parametrization

$$e^{i\theta_1}=\frac{\mathrm{cd}(2\lambda_2)}{\mathrm{cd}(2\lambda_1)},\quad e^{i\theta_2}=-\frac{\mathrm{cd}(2\lambda_1)}{\mathrm{cd}(2\lambda_3)},\quad e^{i\theta_3}=\frac{\mathrm{cd}(2\lambda_3)}{\mathrm{cd}(2\lambda_2)},\quad cd(x)=\frac{\mathrm{cn}(x)}{\mathrm{dn}(x)}$$

$$e^{-i\theta_4}=-\frac{(1+\mathrm{sn}(2\lambda_1))(1-k\,\mathrm{sn}(2\lambda_1))}{\mathrm{cn}(2\lambda_1)\mathrm{dn}(2\lambda_1)},\quad e^{i\theta_5}=\frac{(1+\mathrm{sn}(2\lambda_2))(1-k\,\mathrm{sn}(2\lambda_2))}{\mathrm{cn}(2\lambda_2)\mathrm{dn}(2\lambda_2)},$$

$$e^{i\theta_6}=\frac{(1+\mathrm{sn}(2\lambda_3))(1-k\,\mathrm{sn}(2\lambda_3))}{\mathrm{cn}(2\lambda_3)\mathrm{dn}(2\lambda_3)}$$

$$\alpha_0=\frac{\theta_1+\theta_2+\theta_3-\pi}{2},\quad \alpha_i=\theta_i-\alpha_0\quad t_i=\sqrt{\tan\frac{\alpha_i}{2}}\,.$$

$$\phi(\lambda)=\frac{k'^2\,\mathrm{sn}\lambda}{(\mathrm{cn}\lambda+\mathrm{dn}\lambda)(k\,\mathrm{cn}\lambda+\mathrm{dn}\lambda)}$$

$$t_0=\sqrt{-i\phi(\lambda_-)/\phi(K-\lambda_+)},\quad t_1=\sqrt{i\phi(\lambda_-)\phi(K-\lambda_+)},$$
$$t_2=\sqrt{-i\phi(K-\lambda_-)\phi(\lambda_+)},\quad t_3=\sqrt{i\phi(K-\lambda_-)/\phi(\lambda_+)}$$

where

$$\lambda_\pm=\lambda_1\pm\lambda_2.$$

The weights

$$L^{j;j_2,j_3}_{i;i_2,i_3} = \frac{1}{2} \sum_{n_2,n_3,m_2,m_3=0}^{1} (-)^{i_2 n_2 + i_3 n_3 + j_2 m_2 + j_3 m_3} R^{j,m_2,m_3}_{i,n_2,n_3}$$

Collection of matrices L:

$$(L^0_0)_{2,3} = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \quad (L^1_1)_{2,3} = \begin{pmatrix} b & 0 & 0 & -c \\ 0 & a & -d & 0 \\ 0 & -d & a & 0 \\ -c & 0 & 0 & b \end{pmatrix}, \tag{1}$$

$$a = 1 - t_0 t_1 + t_2 t_3 + t_0 t_1 t_2 t_3 \,, \quad b = 1 + t_0 t_1 - t_2 t_3 + t_0 t_1 t_2 t_3 \,,$$

$$c = 1 + t_0 t_1 + t_2 t_3 - t_0 t_1 t_2 t_3 \,, \quad d = 1 - t_0 t_1 - t_2 t_3 - t_0 t_1 t_2 t_3 \,.$$

$$(L^1_0)_{2,3} = \begin{pmatrix} -a' & 0 & 0 & -d' \\ 0 & -b' & -c' & 0 \\ 0 & c' & b' & 0 \\ d' & 0 & 0 & a' \end{pmatrix}, \quad (L^0_1)_{2,3} = \begin{pmatrix} -b' & 0 & 0 & c' \\ 0 & -a' & d' & 0 \\ 0 & -d' & a' & 0 \\ -c' & 0 & 0 & b' \end{pmatrix}, \tag{2}$$

$$a' = -t_1 t_2 - t_0 t_3 + i t_0 t_2 + i t_1 t_3 \,, \quad b' = -t_1 t_2 - t_0 t_3 - i t_0 t_2 - i t_1 t_3 \,,$$

$$c' = -t_1 t_2 + t_0 t_3 + i t_0 t_2 - i t_1 t_3 \,, \quad d' = -t_1 t_2 + t_0 t_3 - i t_0 t_2 + i t_1 t_3 \,.$$

Parametrization

$$a = \rho_- \,\mathrm{cd}\lambda_-, \quad b = \rho_- \,\mathrm{sn}\lambda_-, \quad c = \rho_-, \quad d = \rho_- \, k\, \mathrm{cd}\lambda_- \,\mathrm{sn}\lambda_-,$$

$$a' = \rho_+ \,\mathrm{cd}\lambda_+, \quad b' = \rho_+ \,\mathrm{sn}\lambda_+, \quad c' = \rho_+, \quad d' = \rho_+ \, k\, \mathrm{cd}\lambda_+ \,\mathrm{sn}\lambda_+,$$

$$\lambda_\pm = \lambda_1 \pm \lambda_2.$$

$$\rho_- = \frac{4(1-\mathrm{sn}\lambda_-)\mathrm{dn}\lambda_-}{\mathrm{cn}\lambda_- \left(\mathrm{cn}\lambda_- \,\mathrm{dn}\lambda_- + (1-\mathrm{sn}\lambda_-)(1+k\,\mathrm{sn}\lambda_-)\right)},$$

$$\rho_+ = -\rho_- \sqrt{\frac{\mathrm{cd}(\lambda_1-\lambda_2)\mathrm{sn}(\lambda_1-\lambda_2)}{\mathrm{cd}(\lambda_1+\lambda_2)\mathrm{sn}(\lambda_1+\lambda_2)}}$$

The tetrahedron equation becomes

$$\sum_{b_1,b_2,b_3} S_{a_1,a_2,a_3}^{b_1,b_2,b_3} (L_{b_1}^{c_1})_{4,5}(\lambda_1,\lambda_2)(L_{b_2}^{c_2})_{4,6}(\lambda_1,\lambda_3)(L_{b_3}^{c_3})_{5,6}(\lambda_2,\lambda_3) =$$

$$\sum_{b_1,b_2,b_3} (L_{a_3}^{b_3})_{5,6}(\lambda_2,\lambda_3)(L_{a_2}^{b_2})_{4,6}(\lambda_1,\lambda_3)(L_{a_1}^{b_1})_{4,5}(\lambda_1,\lambda_2) S_{b_1,b_2,b_3}^{c_1,c_2,c_3}$$

Vacuum vectors and Tetrahedral algebra

Vacuum vectors

$$R\, v\otimes v'\otimes v'' = \lambda\, v\otimes v'\otimes v''.$$

When $\theta_1+\theta_2+\theta_3=\pi$, one can take $\lambda=1$ and $v=v'=v''=(1,0)$.
Applying $v\otimes v\otimes v$ to the TE from the left

$$\sum_{b_1,b_2,b_3} S^{b_1,b_2,b_3}_{a_1,a_2,a_3}(L^{c_1}_{b_1})_{4,5}(\lambda_1,\lambda_2)(L^{c_2}_{b_2})_{4,6}(\lambda_1,\lambda_3)(L^{c_3}_{b_3})_{5,6}(\lambda_2,\lambda_3) =$$
$$\sum_{b_1,b_2,b_3} (L^{b_3}_{a_3})_{5,6}(\lambda_2,\lambda_3)(L^{b_2}_{a_2})_{4,6}(\lambda_1,\lambda_3)(L^{b_1}_{a_1})_{4,5}(\lambda_1,\lambda_2)S^{c_1,c_2,c_3}_{b_1,b_2,b_3}$$

we get

$$L^a_{12}(\lambda_1,\lambda_2)L^b_{1,3}(\lambda_1,\lambda_3)L^c_{2,3}(\lambda_2,\lambda_3) = \sum_{def} S^{def}_{abc}\, L^f_{2,3}(\lambda_2,\lambda_3)L^e_{1,3}(\lambda_1,\lambda_3)L^d_{1,2}(\lambda_1,\lambda_2)$$

where $L^a_{1,2}(\lambda_1,\lambda_2)=(L^a_0)_{1,2}(\lambda_1,\lambda_2)$, etc.

This is the tetrahedral Zamolodchikov algebra used by Korepanov (1993) to calculate the matrix S in the static limit.

THANK YOU