Boundary conditions of discretely holomorphic observables and integrability in lattice models

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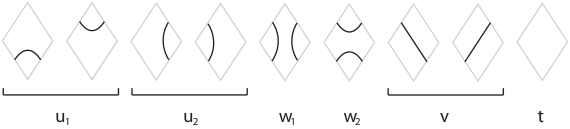
ANZAMP Meeting, Lorne, December 2012

Overview

- The dilute $O(n)$ loop model
- Discretely holomorphic observables
- Integrable solutions
- Boundary conditions
- Solutions for integrable boundary weights
- Conclusion

Dilute $O(n)$ loop model

- Start with a regular, infinite lattice whose faces are rhombi with opening angle α
- A *configuration* is a tiling of the lattice that gives a collection of closed loops



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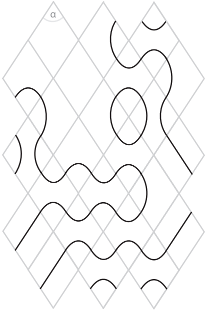
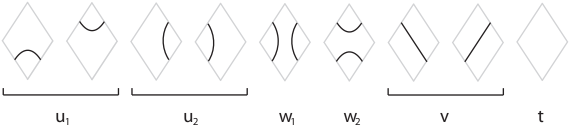




Partition function

$$Z := \sum_G P(G) = \sum_G t^a u_1^b u_2^c w_1^d w_2^e v^f n^N$$

where the sum is over all configurations $P(G)$ and the exponents indicate the number of each type of plaquette or loop.

$n = -2\cos(4\lambda)$ is the weight of a closed loop.

Limits of the model

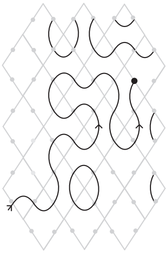
- $n \to -2$: Random Walks
- $n \to 0$: Self-Avoiding Walks
- $n \to 1$: Ising model

Discretely holomorphic observables

Smirnov's *parafermionic* or *discretely holomorphic* observable:

$$F_s(z) = \sum_{\gamma:-\infty\to z} P(\gamma)e^{-isW(\gamma)}$$

- γ is a configuration consisting of an open path from $-\infty$ to z
- $W(\gamma)$ is the winding angle at z
- s is the parafermionic or conformal spin.



Discrete holomorphicity condition

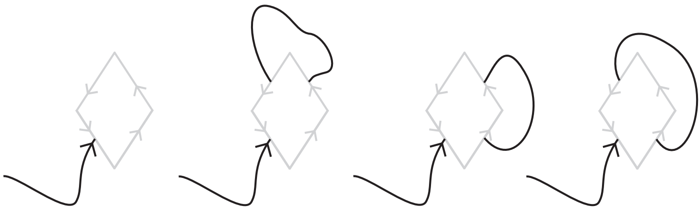
The observable satisfies the following condition:

$$\sum_C F_s(z_i)\Delta z_i = 0$$

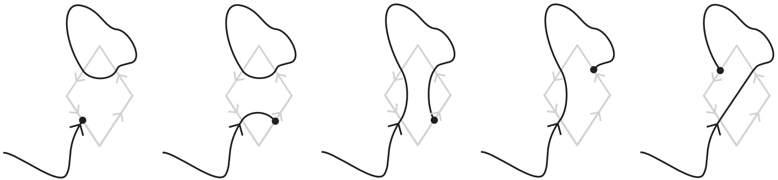
- Equivalent to a discrete form of the Cauchy-Riemann equations
- Holds around an closed contour of the lattice, including around a single face.

Cardy and Ikhlef ('09) looked at what happens if we take the contour around a single face.

There are four disjoint sets of configurations to consider, according to the external connectivity of the edges of the tile:

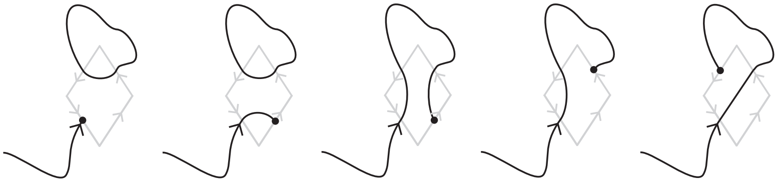


Consider the second of these



Evaluating the discrete contour integral (with $x = \alpha(s-1)$):

$$P(\gamma)(nu_2(x)e^{-i\frac{\pi-\alpha}{2}}e^{-i\frac{\alpha}{2}s}$$



Evaluating the discrete contour integral (with $x = \alpha(s-1)$):

$$P(\gamma)\left(nu_2(x)e^{-i\frac{\pi-\alpha}{2}}e^{-i\frac{\alpha}{2}s} + nw_2(x)e^{i\frac{\pi-\alpha}{2}}e^{i\frac{\alpha}{2}s}\right.$$

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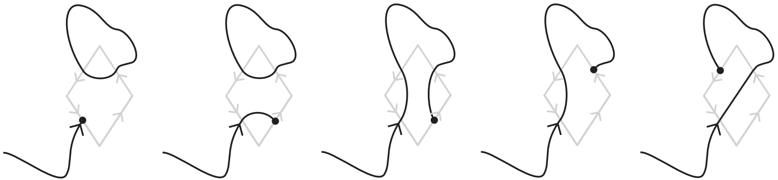

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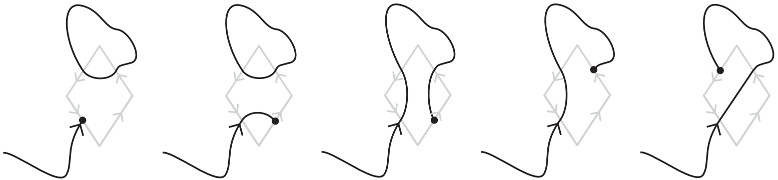
$$\left.-u_1(x)e^{-i\frac{\pi-\alpha}{2}}e^{-i(\frac{\alpha}{2}-\pi)s}\right.$$



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$$-u_1(x)e^{-i\frac{\pi-\alpha}{2}}e^{-i(\frac{\alpha}{2}-\pi)s} - v(x)e^{i\frac{\pi-\alpha}{2}}e^{-i(2\pi-\frac{\alpha}{2})s}\big) = 0$$

This holds around an arbitrary plaquette.

$$-inu_2(x)e^{-i\frac{x}{2}} + inw_2(x)e^{i\frac{x}{2}} + iw_1(x)e^{i\frac{x}{2}}$$
$$+iu_1(x)e^{i\pi s}e^{-i\frac{x}{2}} - iv(x)e^{-i2\pi s}e^{i\frac{x}{2}} = 0$$

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- Solve for the Boltzmann weights

$$
\begin{aligned}
t(x) &= \sin(x)\sin(3\lambda - x) + \sin(2\lambda)\sin(3\lambda) \\
u_1(x) &= \sin(2\lambda)\sin(3\lambda - x) \\
u_2(x) &= \sin(2\lambda)\sin(x) \\
v(x) &= \sin(x)\sin(3\lambda - x) \\
w_1(x) &= \sin(2\lambda - x)\sin(3\lambda - x) \\
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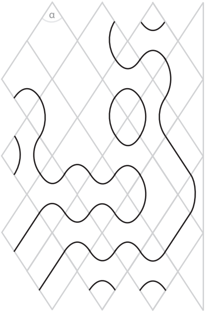
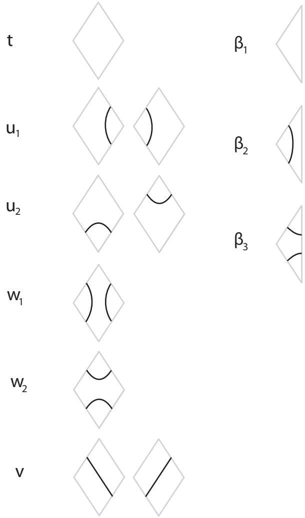
These are the Yang-Baxter integrable weights.

Boundaries

Now introduce a single boundary on the right-hand side of the domain as well as three boundary plaquettes.





Boundary conditions

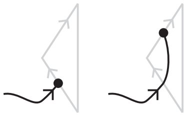
What boundary condition should $F_s(z)$ satisfy?

For *diagonal boundaries* ($\beta_3 = 0$) the following condition seems to work (Beaton et al. '11, Ikhlef '12):

$$Re\left(\sum_C F(z_i)\Delta z_i\right) = 0.$$

with the contour sum taken along the two edges of a boundary tile.

Diagonal boundaries



Taking the real part of the contour sum

$$Re[P(\gamma)e^{-is\frac{\pi}{2}}\left(e^{ix}\beta_1(x) - e^{-ix}\beta_2(x)\right)] = 0$$

we find the solutions

$$\beta_1(x) = c_1 \sin(x + \frac{3\lambda}{2}), \qquad \beta_2(x) = -c_1 \sin(x - \frac{3\lambda}{2}).$$

These are also solutions of the reflection Yang-Baxter equation (Iklhef '12, Batchelor and Yung '95).

Diagonal boundaries

We could instead require

$$Im\left(\sum_{C} F(z_i)\Delta z_i\right) = 0$$

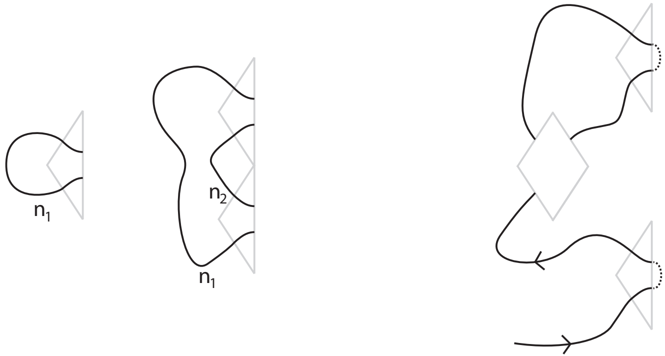
Solving the resulting linear equation gives

$$\beta_1(x) = c_1\cos(x + \frac{3\lambda}{2}), \qquad \beta_2(x) = c_1\cos(x - \frac{3\lambda}{2}).$$

Another, independent set of solutions of the reflection equation (Batchelor and Yung '95).

More general boundaries

Including the β_3 boundary tile allows for two additional types of loops, with weights n_1, n_2:



- n_1 (n_2) loops whose upper terminal starts on the upper (lower) edge of a β_3 plaquette
- Additional configurations $\rightarrow$ need to modify $F_s(z)$

Boundary observable

The following observable is discretely holomorphic

$$F_s(z) = \sum_{\gamma:-\infty\to z} P(\gamma) e^{-iW(\gamma)s} \qquad \underbrace{q^{t_1}\bar{q}^{t_2} n_3^{N_3}}_{\text{boundary interactions}}$$

- $\bar{q}$ (q), if the boundary defect has passed through a β_3 tile in a clockwise (counter-clockwise) sense ($t_i = 0, 1$)
- New loop weight n_3 - the open loop segment can pass between two top or bottom edges of β_3 boundary plaquettes
- From requiring discrete holomorphicity and also that n_3 be real we obtain the following

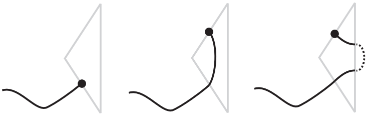
$$q = -\frac{n_1}{2\cos(4\lambda - \eta)} e^{i\eta}, \qquad n_3 = n_1 \frac{\sin(4\lambda)}{\sin(2\eta - 8\lambda)}$$

where

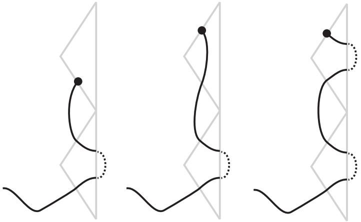
$$n = -2\cos(4\lambda)$$

and η is an arbitrary parameter.

Boundary configurations



$$e^{ix}e^{-is\frac{\pi}{2}}\beta_1(x) - e^{-is\frac{\pi}{2}}e^{-ix}\beta_2(x) - qe^{-is\frac{\pi}{2}}e^{-ix}\beta_3(x)$$



$$qe^{ix}e^{-is\frac{\pi}{2}}\beta_1(x) - qe^{-is\frac{\pi}{2}}e^{-ix}\beta_2(x) - qn_2e^{-is\frac{\pi}{2}}e^{-ix}\beta_3(x)$$

Requiring the first boundary condition to be satisfied

$$Re\left(\sum_{C} F(z_i)\Delta z_i\right) = 0$$

gives two linear equations in β_1, β_2 and β_3.

$$\begin{aligned}
\beta_1(x) &= n_1 + n_2\cos(4\lambda) + n_3\cos(2x-\lambda)\\
\beta_2(x) &= n_1\cos(2x) + n_2\cos(2x-4\lambda) + n_3\cos(\lambda)\\
\beta_3(x) &= -2\sin(4\lambda)\sin(2x)
\end{aligned}$$

where $n_3 = \sqrt{n_1^2 + n_2^2 - n n_1 n_2}$.

Alternatively, requiring the second boundary condition to be satisfied

$$Im\left(\sum_{C} F(z_i)\Delta z_i\right) = 0$$

$\Rightarrow$

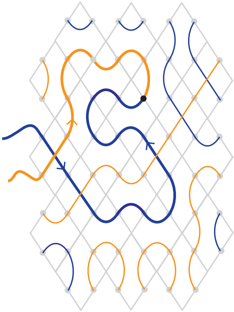
$$\begin{aligned}
\beta_1(x) &= n_1 + n_2\cos(4\lambda) - n_3\cos(2x-\lambda) \\
\beta_2(x) &= n_1\cos(2x) + n_2\cos(2x-4\lambda) - n_3\cos(\lambda) \\
\beta_3(x) &= -2\sin(4\lambda)\sin(2x)
\end{aligned}$$

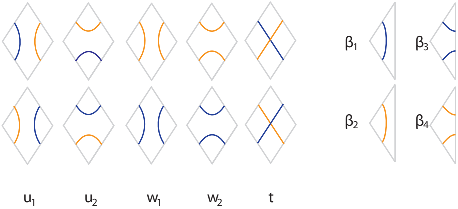
Both sets are solutions of the boundary Yang-Baxter equation. In the case $n_2 = 1$ the solutions simplify to those of the blobbed $O(n)$ model of Dubail, Jacobsen and Saleur ('09).

$C_2^{(1)}$ model

A parafermionic observable for the $C_2^{(1)}$ model was found by Cardy and Ikhlef ('09). A similar observable exists when the domain contains a boundary.

$$F_s(z) = \sum_{\gamma_1, \gamma_2 : -\infty \to z} P(\gamma_1, \gamma_2) e^{-i(W(\gamma_1)+W(\gamma_2))s} q^{t_1} \bar{q}^{t_2} n_3^{N_3}$$





Integrable weights

Diagonal boundaries plaquettes:

$$\beta_1(x) = \beta_2(x), \qquad \beta_1(x) = -\beta_2(x).$$

General boundary plaquettes:

$$\begin{aligned}
\beta_1(x) &= n_1 \cos(x - 4\lambda - 4\lambda_1) \\
\beta_2(x) &= \beta_1(x) \\
\beta_3(x) &= 2\sin(4\lambda_1)\sin(x) \\
\beta_4(x) &= \beta_3(x).
\end{aligned}$$

and

$$\begin{aligned}
\beta_1(x) &= n_1 \sin(x - 4\lambda - 4\lambda_1) \\
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\end{aligned}$$

where $n = -2\cos(4\lambda)$, $n_1 = -2\rho\cos(2\lambda_1)$.

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- Rigorous proofs of the location of critical points? (Duminil-Copin and Smirnov '10, Beaton et al. '11)
- A better understanding of the connection between integrability and discrete holomorphicity (Alam, Batchelor '12, Fendley '12)