Superintegrability in a non-conformally-flat space

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What is a superintegrable system?

Higher Symmetries of the Laplacian

Extending the superintegrable TTW systems to higher dimensions

Superintegrability suggests a non-standard conformally covariant Laplacian

Consider H, a natural Hamiltonian on a $2n$-dimensional phase space

$$H = \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(x_1, \ldots, x_n).$$

H is Liouville integrable means there are n functions of the phase space

$$L_0 = H, L_1, \ldots, L_{n-1},$$

that are

- functionally, independent
- globally defined,
- Poisson commuting: $\{L_i, L_j\} = 0$ for $i = 0 \ldots n-1$.

H is superintegrable means that H is integrable and additionally there are further functionally independent constants up to a maximum of $2n-1$. That is,

$$\{H, L_i\} = 0 \qquad \text{for } i = 0 \ldots 2n-2.$$

Here consider only constants polynomial in the momenta. Eg, second order constants:

$$L_i = \sum_{j,k=1}^{n} a_{(i)}^{jk} p_j p_k + W_{(i)}(x_1, \ldots, x_n).$$

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What is a Superintegrable Systems?

- The trajectories can be found algebraically. They must lie on the intersection of the level sets of the $2n - 1$ constants.
- The Kepler-Coulomb and isotropic oscillator systems are the best known examples.
- For quantum superintegrability momenta and the Poisson brackets replaced by differential operators and the operator commutators.
- Classical and quantum superintegrable systems typically have a Poisson or commutator algebra that closes polynomially.
- Superintegrable systems with all second order symmetries are multi-separable and provide information on special functions.
- Connections with Quasi-Exactly-Solvable systems.

The Hamiltonian

$$H = p_x^2 + p_y^2 + \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$$

has two constants that are second order polynomials in the momenta

$$L_1 = p_x^2 + \alpha x^2 + \frac{\beta}{x^2}, \qquad L_2 = (xp_y - yp_x)^2 + \beta\frac{y^2}{x^2} + \gamma\frac{x^2}{y^2}.$$

$$\{H, L_1\} = \{H, L_2\} = 0.$$

With $R = \{L_1, L_2\}$ the Poisson algebra closes quadratically.

$$\begin{aligned}
\{R, L_1\} &= 8L_1^2 - 8HL_1 + 16\alpha L_2, \\
\{R, L_2\} &= -16L_1L_2 + 8HL_2 - 16(\beta+\gamma)L_1 + 16H\beta, \\
R^2 &= -16L_1^2L_2 + 16HL_1L_2 - 16\alpha L_2^2 - 16(\beta+\gamma)L_2^2 + 32\beta HL_1 \\
&\quad -16\beta H^2 + 16\alpha\beta\gamma.
\end{aligned}$$

Some examples...

- Oscillators with rational frequency ratios

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \omega_1^2 x^2 + \omega_2^2 y^2 \,, \quad \frac{\omega_1}{\omega_2} \in \mathbb{Q}$$

- Calogero-Moser (Wojciechovski, 1983)

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j=1}^{n} \frac{1}{(x_i - x_j)^2}$$

- Toda lattice (Agrotis, Damianou, Sophocleous, 2005)

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{x_i - x_{i+1}}$$

- A non-separable system (Post and Winternitz, 2011)

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\alpha y}{x^{2/3}}$$

Homogeneous polynomials in the momenta correspond to symmetric contravariant tensors.

If H_0 and L_0 are the leading terms of H and L as polynomials in the momenta, then

$$\{H_0, L_0\} = 0 \qquad \Longleftrightarrow \qquad [g, K] = 0$$

where [,] is the Schouten bracket and

$$g = g^{ij} \frac{\partial}{\partial x_i} \odot \frac{\partial}{\partial x_j} \qquad \text{and} \qquad K = a^{i_1 \cdots i_r} \frac{\partial}{\partial x_{i_1}} \odot \cdots \odot \frac{\partial}{\partial x_{i_r}}.$$

K is a second rank (symmetric) Killing tensor.

The leading terms (symbol) of a differential operator symmetry of the (conformal) Laplacian is a (conformal) Killing tensor.

Do all (conformal) Killing tensors correspond to a symmetry of the (conformal) Laplacian?

Tremblay, Turbiner and Winternitz (2009) introduced a family of potentials

$$V_{TTW} = \alpha r^2 + \frac{\beta}{r^2 \cos^2(k\theta)} + \frac{\gamma}{r^2 \sin^2(k\theta)}.$$

For $k = 1, 2, 3$ these were known systems (Smorodinsky-Winternitz, rational BC_2 model, Wolfes model).

For all $k \in \mathbb{N}^+$, Tremblay, Turbiner and Winternitz

- showed that the quantum systems are *exactly solvable* and the classical systems have closed trajectories.
- demonstrated superintegrability for $k = 1, 2, 3, 4$.
- conjectured superintegrability for rational k.

The classical and quantum TTW systems are superintegrable for all positive rational parameter values (KMP 2009, KKM 2010).

These results have been extended to other families, eg

$$H = \cosh^2\psi \left(p_\psi^2 + p_\varphi^2 + \frac{\alpha}{\cos^2 k\varphi} + \frac{\beta}{\sin^2 k\varphi} + \frac{\gamma}{\sinh^2\psi} \right).$$

The methods used allow explicit calculation of the symmetry algebra which closes polynomially.

In the following I will describe some extensions to higher dimensions.

The classical TTW Hamiltonian for $k = 1$ can be written in Cartesian coordinates as

$$H_{TTW} = p_x^2 + p_y^2 + \alpha(x^2 + y^2) + \frac{\beta_1}{x^2} + \frac{\beta_2}{y^2}$$

and in polar coordinates as

$$H_{TTW} = p_r^2 + \alpha r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{\beta_1}{\cos^2\theta} + \frac{\beta_2}{\sin^2\theta}\right)$$

A natural generalization to 3 dimensions is

$$H = p_x^2 + p_y^2 + p_z^2 + \alpha(x^2 + y^2 + z^2) + \frac{\beta_1}{z^2} + \frac{\beta_2}{x^2} + \frac{\beta_3}{y^2}$$

or in polar coordinates,

$$H = p_r^2 + \alpha r^2 + \frac{1}{r^2}\left(p_{\theta_1}^2 + \frac{\beta_1}{\cos^2\theta_1} + \frac{1}{\sin^2\theta_1}\left(p_{\theta_2}^2 + \frac{\beta_2}{\cos^2\theta_2} + \frac{\beta_3}{\sin^2\theta_2}\right)\right)$$

This pattern will continue in higher dimensions.

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This pattern will continue in higher dimensions.

The classical TTW Hamiltonian for arbitrary k is written in polar coordinates as

$$H_{TTW} = p_r^2 + \alpha r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{\beta_1}{\cos^2(k\theta)} + \frac{\beta_2}{\sin^2(k\theta)}\right)$$

and so we generalize this as

$$H = p_r^2 + \alpha r^2 + \frac{1}{r^2}\left(p_{\theta_1}^2 + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{1}{\sin^2(k_1\theta_1)}\left(p_{\theta_2}^2 + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{\beta_3}{\sin^2(k_2\theta_2)}\right)\right)$$

and so on to higher dimensions.

Inside each pair of brackets is a constant of the motion.

Such systems are superintegrable.

We now consider a 4-dimensional generalization of the TTW system.

$$
\begin{aligned}
H = L_1 &= p_r^2 + \alpha r^2 + \frac{L_2}{r^2} \\
L_2 &= p_{\theta_1}^2 + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_3}{\sin^2(k_1\theta_1)} \\
L_3 &= p_{\theta_2}^2 + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_4}{\sin^2(k_2\theta_2)} \\
L_4 &= p_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)}.
\end{aligned}
$$

This system is superintegrable for all positive rational k_1, k_2 and k_3.

The corresponding metric

$$g = e_0 \otimes e_0 + e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3,$$

$$e_0 = dr, \quad e_1 = r d\theta_1, \quad e_2 = r \sin(k_1\theta_1) d\theta_2, \quad e_3 = r \sin(k_1\theta_1) \sin(k_2\theta_2) d\theta_3.$$

is not conformally flat since

$$W_{abcd} W^{abcd} = \frac{4(k_1^2 - k_2^2)^2}{3r^4 \sin^4(k_1\theta_1)}.$$

W_{abcd} is the Weyl conformal curvature tensor.

The metric g is conformally flat if and only if $k_1^2 = k_2^2$.

Next, illustrate the superintegrability with $k_1 = 2, k_2 = k_3 = 1$. The general case is similar.

As an example, with $k_1, k_2, k_3 = 2, 1, 1$, we find two additional quartic and a cubic constant of the motion

$$\begin{aligned}
L_1'' &= \left(H - \frac{2L_2}{r^2}\right)\frac{\sin(4\theta_1)}{r} p_{\theta_1} p_r \\
&\quad + \frac{2(L_2\cos(4\theta_1) + L_3 - \beta_1)}{r^2}\ p_r^2 - \frac{1}{4}(H^2 - \alpha L_2)\cos(4\theta_1), \\
L_2'' &= 2(L_3\cos(2\theta_2) + L_4 - \beta_2)\cot(2\theta_1)\sin(2\theta_2) p_{\theta_1} p_{\theta_2} \\
&\quad + ((\beta_2 - L_3 - L_4)^2 - 4L_3L_4)\mathrm{cosec}^2(2\theta_1) \\
&\quad - \sin^2(2\theta_2)(2L_3\mathrm{cosec}^2(2\theta_1) + \beta_1 - L_2 - L_3)p_{\theta_2}^2 \\
L_3'' &= 2(L_4\cos(2\theta_3) + \beta_4 - \beta_3)\cot(\theta_2)p_{\theta_2} \\
&\quad - (2L_4\mathrm{cosec}^2(\theta_2) + \beta_2 - L_3 - L_4)\sin(2\theta_3)p_{\theta_3}
\end{aligned}$$

Setting $\alpha = \beta_1 = \beta_2 = \beta_3 = 0$ gives higher rank Killing tensors independent of the rank 2 Killing tensors.

The quantum version has four mutually commuting differential operators,

$$\begin{aligned} H &= \partial_r^2 + \frac{3}{r}\partial_r - \omega^2 r^2 + \frac{L_1}{r^2} \\ L_1 &= \partial_{\theta_1}^2 + 2k_1 \cot(k_1\theta_1)\partial_{\theta_1} + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_2}{\sin^2(k_1\theta_1)} \\ L_2 &= \partial_{\theta_2}^2 + k_2 \cot(k_2\theta_2)\partial_{\theta_2} + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_3}{\sin^2(k_2\theta_2)} \\ L_3 &= \partial_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)} \end{aligned}$$

The Hamiltonian is of the form $H = \nabla^2 + V_0$ where ∇^2 is the Laplacian on a four-dimensional manifold with metric

$$g = e_0 \otimes e_0 + e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3,$$

$$e_0 = dr, \quad e_1 = r d\theta_1, \quad e_2 = r\sin(k_1\theta_1) d\theta_2, \quad e_3 = r\sin(k_1\theta_1)\sin(k_2\theta_2) d\theta_3.$$

The equation $H\Psi = E\Psi$ is separable in the coordinates $r, \theta_1, \theta_2, \theta_3$ but for reasons that will be 'explained', we add some extra terms.

$$
\begin{aligned}
H = L_0 &= \partial_r^2 + \frac{3}{r}\partial_r - \omega^2 r^2 + \frac{L_1}{r^2} + \frac{1-k_1^2}{r^2} \\
L_1 &= \partial_{\theta_1}^2 + 2k_1\cot(k_1\theta_1)\partial_{\theta_1} + \frac{\beta_1}{\cos^2(k_1\theta_1)} + \frac{L_2}{\sin^2(k_1\theta_1)} + \frac{k_1^2 - k_2^2}{4\sin^2(k_1\theta_1)} \\
L_2 &= \partial_{\theta_2}^2 + k_2\cot(k_2\theta_2)\partial_{\theta_2} + \frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{L_3}{\sin^2(k_2\theta_2)} \\
L_3 &= \partial_{\theta_3}^2 + \frac{\beta_3}{\cos^2(k_3\theta_3)} + \frac{\beta_4}{\sin^2(k_3\theta_3)}
\end{aligned}
$$

$H\Psi = E\Psi$ remains separable with the additional terms.

For the metric g, the scalar curvature is

$$\mathcal{R} = -\frac{6}{r^2} + k_1^2\left(\frac{6}{r^2} - \frac{2}{r^2\sin^2(k_1\theta_1)}\right) + \frac{2k_2^2}{r^2\sin^2(k_1\theta_1)},$$

and with Weyl conformal tensor W_{abcd}, if we define

$$\mathcal{W} = \sqrt{3W_{abcd}W^{abcd}} = \frac{2(k_1^2 - k_2^2)}{r^2\sin^2(k_1\theta_1)}.$$

then

$$H = \nabla^2 + V_0 - \frac{1}{6}\mathcal{R} - \frac{1}{24}\mathcal{W}.$$

The metric g is conformally flat if and only if $k_1^2 = k_2^2$.

We look for solutions of the form

$$H\Psi = E\Psi, \qquad \Psi = \Psi_0(r)\Psi_1(\theta_1)\Psi_2(\theta_2)\Psi_3(\theta_3),$$

and

$$L_3\Psi_3 = \ell_3\Psi_3, \qquad L_2\Psi_2\Psi_3 = \ell_2\Psi_2\Psi_3, \qquad L_1\Psi_1\Psi_2\Psi_3 = \ell_1\Psi_1\Psi_2\Psi_3.$$

For the angular equations we will find the equation

$$u'' + \left(\frac{\frac{1}{4}-\alpha^2}{\sin^2 y} + \frac{\frac{1}{4}-\beta^2}{\cos^2 y} + (2n+\alpha+\beta+1)^2\right) u = 0.$$

which has solution

$$u(y) = (\sin y)^{\alpha+\frac{1}{2}}(\cos y)^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos 2y)$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi function.

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For the angular equations we will find the equation

$$u'' + \left(\frac{\frac{1}{4} - \alpha^2}{\sin^2 y} + \frac{\frac{1}{4} - \beta^2}{\cos^2 y} + (2n + \alpha + \beta + 1)^2 \right) u = 0.$$

which has solution

$$u(y) = (\sin y)^{\alpha + \frac{1}{2}} (\cos y)^{\beta + \frac{1}{2}} P_n^{(\alpha,\beta)}(\cos 2y)$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi function.

With the replacements

$$\beta_3 = k_3^2 \left(\frac{1}{4} - a_4^2 \right), \qquad \beta_4 = k_3^2 \left(\frac{1}{4} - a_3^2 \right),$$

the separated θ_3 equation is

$$L_3 \Psi_3(\theta_3) = \Psi_3''(\theta_3) + \left(\frac{k_3^2 \left(\frac{1}{4} - a_4^2 \right)}{\cos^2(k_3\theta_3)} + \frac{k_3^2 \left(\frac{1}{4} - a_3^2 \right)}{\sin^2(k_3\theta_3)} \right) \Psi_3(\theta_3) = \ell_3 \Psi_3(\theta_3)$$

which has solutions

$$\Psi_{3,n_3}^{a_3,a_4}(\theta_3) = (\sin(k_3\theta_3))^{a_3+\frac{1}{2}} (\cos(k_3\theta_3))^{a_4+\frac{1}{2}} P_{n_3}^{(a_3,a_4)}(\cos(2k_3\theta_3))$$

and eigenvalues

$$L_3 \Psi_{3,n_3}^{a_3,a_4}(\theta_3) = \ell_3 \Psi_{3,n_3}^{a_3,a_4}(\theta_3), \qquad \ell_3 = -k_3^2(2n_3 + a_3 + a_4 + 1)^2.$$

The separated θ_2 equation $L_2\Psi_2(\theta_2) = \ell_2\Psi_2(\theta_2)$ is

$$\Psi_2''(\theta_2) + k_2\cot(k_2\theta_2)\Psi_2'(\theta_2) + \left(\frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{\ell_3}{\sin^2(k_2\theta_2)}\right)\Psi_2(\theta_2) = \ell_2\Psi_2(\theta_2)$$

which we transform with

$$\Psi_2(\theta_2) = (\sin(k_2\theta_2))^{-\frac{1}{2}}\psi_2(\theta_2)$$

to absorb the first derivative term to give

$$\psi_2''(\theta_2) + \left(\frac{\beta_2}{\cos^2(k_2\theta_2)} + \frac{\ell_3 + \frac{1}{4}k_2^2}{\sin^2(k_2\theta_2)} + \frac{1}{4}k_2^2 - \ell_2\right)\psi_2(\theta_2) = 0$$

and we make the replacements

$$\beta_2 = k_2^2\left(\frac{1}{4} - a_2^2\right), \qquad \ell_3 + \frac{1}{4}k_2^2 = k_2^2\left(\frac{1}{4} - A_2^2\right).$$

$$\boxed{\ell_3 = -k_3^2(2n_3 + a_3 + a_4 + 1)^2 \quad \Rightarrow \quad A_2 = \frac{k_3}{k_2}(2n_3 + a_3 + a_4 + 1)}$$

The separated θ_2 equation becomes

$$\psi_2''(\theta_2) + \left(\frac{k_2^2 \left(\frac{1}{4} - a_2^2\right)}{\cos^2(k_2\theta_2)} + \frac{k_2^2 \left(\frac{1}{4} - A_2^2\right)}{\sin^2(k_2\theta_2)} + \frac{k_2^2}{4} - \ell_2 \right) \psi_2(\theta_2) = 0$$

where

$$\frac{\mathbf{k}_2^2}{\mathbf{4}} - \ell_2 = \mathbf{k}_2^2(\mathbf{2n}_2 + \mathbf{a}_2 + \mathbf{A}_2 + \mathbf{1})^2$$

and has solution

$$\Psi_{2,n_2}^{A_2,a_2}(\theta_2) = (\sin(k_2\theta_2))^{A_2}(\cos(k_2\theta_2))^{a_2+\frac{1}{2}} P_{n_2}^{(A_2,a_2)}(\cos(2k_2\theta_2)).$$

The separated θ_1 equation $L_1\Psi_1(\theta_1) = \ell_1\Psi_1(\theta_1)$ is

$$\Psi_1''(\theta_1)+2k_1\cot(k_1\theta_1)\Psi_1'(\theta_1)+\left(\frac{\beta_1}{\cos^2(k_1\theta_1)}+\frac{\ell_2+\frac{1}{4}(\mathbf{k}_1^2-\mathbf{k}_2^2)}{\sin^2(k_1\theta_1)}\right)\Psi_1(\theta_1)=\ell_1\Psi_1(\theta_1),$$

which we transform with

$$\Psi_1(\theta_1) = (\sin(k_1\theta_1))^{-1}\psi(\theta_1)$$

to absorb the first order term to give

$$\psi_1''(\theta_1)+\left(\frac{\beta_1}{\cos^2(k_1\theta_1)}+\frac{\ell_2+\frac{1}{4}(\mathbf{k}_1^2-\mathbf{k}_2^2)}{\sin^2(k_1\theta_1)}+k_1^2-\ell_1\right)\psi_1(\theta_1)=0$$

and we make the replacements

$$\beta_1 = k_1^2\left(\frac{1}{4}-a_1^2\right), \qquad \ell_2+\frac{1}{4}(\mathbf{k}_1^2-\mathbf{k}_2^2) = k_1^2\left(\frac{1}{4}-A_1^2\right).$$

On the last slide we had

$$\ell_2 + \frac{1}{4}(\mathbf{k}_1^2 - \mathbf{k}_2^2) = k_1^2\left(\frac{1}{4} - A_1^2\right)$$

which along with

$$\frac{\mathbf{k}_2^2}{4} - \ell_2 = \mathbf{k}_2^2(2\mathbf{n}_2 + \mathbf{a}_2 + \mathbf{A}_2 + 1)^2$$

gives

$$A_1 = \frac{k_2}{k_1}(2n_2 + A_2 + a_2 + 1).$$

Without the $\frac{1}{4}(\mathbf{k}_1^2 - \mathbf{k}_2^2)$ term, we would have had

$$A_1 = \sqrt{\frac{1}{4}\left(1 - \frac{k_2^2}{k_1^2}\right) + \frac{k_2^2}{k_1^2}(2n_2 + a_2 + A_2 + 1)^2}.$$

The separate θ_1 equation becomes

$$\partial_{\theta_1}^2 \psi_1(\theta_1) + \left(\frac{k_1^2 \left(\frac{1}{4} - a_1^2\right)}{\cos^2(k_1\theta_1)} + \frac{k_1^2 \left(\frac{1}{4} - A_1^2\right)}{\sin^2(k_1\theta_1)} + k_1^2 - \ell_1 \right) \psi_1(\theta_1) = 0$$

where

$$\mathbf{k_1^2 - \ell_1 = k_1^2(2n_1 + A_1 + a_1 + 1)^2}.$$

and has solutions

$$\Psi_{1,n_1}^{A_1,a_1}(\theta_1) = (\sin(k_1\theta_1))^{A_1-\frac{1}{2}} (\cos(k_1\theta_1))^{a_1+\frac{1}{2}} P_{n_1}^{(A_1,a_1)}(\cos(2k_1\theta_1))$$

Finally, the separated radial equation is

$$H\Psi_0(r) = \partial_r^2\Psi_0(r) + \frac{3}{r}\partial_r\Psi_0(r) + \left(-\omega^2 r^2 + \frac{\ell_1 \mathbf{-k_1^2+1}}{r^2}\right)\Psi_0(r) = E\Psi_0(r).$$

We remove the first order terms with the transformation

$$\Psi_0(r) = r^{-\frac{3}{2}}\psi_0(r)$$

to give

$$\partial_r^2\psi_0(r) + \left(-\omega^2 r^2 + \frac{\frac{1}{4}\mathbf{-k_1^2}+\ell_1}{r^2} - E\right)\psi_0(r) = 0.$$

Now,

$$u''(x) + \left(-x^2 + \frac{\frac{1}{4}\mathbf{-A_0^2}}{x^2} + 4n + 2A_0 + 2\right)u(x) = 0$$

has solution

$$u(x) = e^{-\frac{x^2}{2}}x^{A_0+\frac{1}{2}}L_n^{(A_0)}(x^2),$$

where $L_n^{(A_0)}(x)$ is a Laguerre function.

We needed

$$A_0^2 = k_1^2 - \ell_1$$

and we already have

$$\mathbf{k}_1^2 - \ell_1 = \mathbf{k}_1^2(2\mathbf{n}_1 + \mathbf{A}_1 + \mathbf{a}_1 + 1)^2$$

so

$$\Psi_{0,n_0}^{A_0}(r) = e^{-\frac{\omega r^2}{2}} r^{A_0 - 1} L_{n_0}^{(A_0)}(\omega r^2)$$

where $L_n^A(x)$ is a Laguerre function and

$$A_0 = k_1(2n_1 + a_1 + A_1 + 1), \qquad E = -\omega(4n_0 + 2A_0 + 2).$$

Now, putting these together

$$\begin{aligned} E &= -\omega(4n_0 + 2A_0 + 2) \\ A_0 &= k_1(2n_1 + A_1 + a_1 + 1) \\ A_1 &= \frac{k_2}{k_1}(2n_2 + A_2 + a_2 + 1) \\ A_2 &= \frac{k_3}{k_2}(2n_3 + a_3 + a_4 + 1) \end{aligned}$$

we find

$$E = -2\omega(2n_0 + 2k_1 n_1 + 2k_2 n_2 + 2k_3 n_3 + k_1 a_1 + k_2 a_2 + k_3 a_3 + k_3 a_4 + k_1 + k_2 + k_3 + 1)$$

for a solution of the form

$$\Psi_{n_0,n_1,n_2,n_3} = \Psi_{0,n_0}^{A_0}(r)\Psi_{1,n_1}^{A_1,a_1}(\theta_1)\Psi_{2,n_2}^{A_2,a_2}(\theta_2)\Psi_{3,n_3}^{a_3,a_4}(\theta_3).$$

Our aim is to use special function identities to raise and lower the n_i while preserving E and produce new operators commuting with H.

Some examples...

$$K_{0\ n_0}^{+\ A_0} = \frac{1 - A_0}{r}\partial_r + (2n_0 + A_0 + 1)\omega + \frac{1 - A_0^2}{r^2}$$

$$K_{0\ n_0}^{-\ A_0} = \frac{1 + A_0}{r}\partial_r + (2n_0 + A_0 + 1)\omega + \frac{1 - A_0^2}{r^2}$$

These raise and lower n_0 and A_0 simultaneously.

$$K_{0\ n_0}^{+\ A_0}\Psi_{n_0}^{A_0} = -2(n_0 + 1)(n_0 + A_0)\Psi_{n_0+1}^{A_0-2}$$

$$K_{0\ n_0}^{-\ A_0}\Psi_{n_0}^{A_0} = -2\omega^2\Psi_{n_0-1}^{A_0+2}$$

For the angular functions, we can use Jacobi function identities to make operators that raise and lower n alone.

$$
\begin{aligned}
J_n^+ &= -\frac{(N+1)\sin(2k\theta)}{2k}\partial_\theta - \frac{1}{2}\Big((N+1)(N+1-c-d)\cos(2k\theta) \\
&\quad -(N+1)(c-d)+a^2-b^2\Big) \\
J_n^- &= \frac{(N-1)\sin(2k\theta)}{2k}\partial_\theta - \frac{1}{2}\Big((N-1)(N-1+c+d)\cos(2k\theta) \\
&\quad +(N-1)(c-d)+a^2-b^2\Big)
\end{aligned}
$$

$$
N = 2n+a+b+1, \qquad \Theta_n^{(a,b)} = \sin^{a+c}(k\theta)\cos^{b+d}(k\theta)P_n^{(a,b)}\cos(2k\theta)
$$

$$
\begin{aligned}
J_n^+\Theta_n^{(a,b)} &= -2(n+1)(n+a+b+1)\Theta_{n+1}^{(a,b)} \\
J_n^-\Theta_n^{(a,b)} &= -2(n+a)(n+b)\Theta_{n-1}^{(a,b)}
\end{aligned}
$$

We can also use Jacobi function identities to make operators that raise and lower n and a simultaneously.

$$\begin{aligned} K_n^{+\,a} &= -\frac{(1-a)\cos(k\theta)}{k\sin(k\theta)}\partial_\theta - 2(n(n+a+b+1)+a(a+b)) \\ &\quad -(1-a)(a+c+b+d) - \frac{(1-a)(a-c)}{\sin^2(k\theta)} \\ K_n^{-\,a} &= -\frac{(1+a)\cos(k\theta)}{k\sin(k\theta)}\partial_\theta - 2n(n+a+b+1) \\ &\quad -(1+a)(a+c+b+d) + \frac{(1+a)(a+c)}{\sin^2(k\theta)} \end{aligned}$$

Again, for

$$\Theta_n^{(a,b)} = \sin^{a+c}(k\theta)\cos^{b+d}(k\theta)P_n^{(a,b)}\cos(2k\theta)$$

we have

$$\begin{aligned} K_n^{+\,a}\Theta_n^{(a,b)} &= 2(n+1)(n+a)\Theta_{n+1}^{(a-2,b)} \\ K_n^{-\,a}\Theta_n^{(a,b)} &= 2(n+a+b+1)(n+b)\Theta_{n-1}^{(a+2,b)} \end{aligned}$$

For $k_1 = p_1/q_1$ with $\gcd(p_1, q_1) = 1$ the operator

$$\Xi^{+}_{01} = \underbrace{K^{-\,A_0+2(p_1-1)}_{0\;n_0-(p_1-1)} \cdots K^{-\,A_0}_{0\;n_0}}_{p_1 \text{ terms}} \underbrace{J^{+}_{1\;n_1+q_1-1} \cdots J^{+}_{1\;n_1}}_{q_1 \text{ terms}}$$

which has the effect on a basis function of

$$n_0 \to n_0 - p_1, \quad n_1 \to n_1 + q_1, \quad A_0 \to A_0 + 2p_1.$$

and so

$$E = -2\omega(2n_0 + 2k_1 n_1 + \cdots) \to -2\omega(2(n_0 - p_1) + 2k_1(n_1 + q_1) + \cdots)$$
$$= -2\omega(2n_0 + 2k_1 n_1 + \cdots),$$

that is, E is unchanged. A similar lowering operator is

$$\Xi^{-}_{01} = \underbrace{K^{+\,A_0-2(p_1-1)}_{0\;n_0+(p_1-1)} \cdots K^{+\,A_0}_{0\;n_0}}_{p_1 \text{ terms}} \underbrace{J^{-}_{1\;n_1-(q_1-1)} \cdots J^{-}_{1\;n_1}}_{q_1 \text{ terms}}.$$

This is exactly like the original TTW.

For $k_2/k_1 = p_2/q_2$ with $\gcd(p_2, q_2) = 1$ the operator

$$\Xi^+_{12} = \underbrace{K^{-\,A_1+2(p_2-1)}_{1\ n_1-(p_2-1)} \cdots K^{-\,A_1}_{1\ n_1}}_{p_2 \text{ terms}} \underbrace{J^+_{2\ n_2+q_2-1} \cdots J^+_{2\ n_2}}_{q_2 \text{ terms}}$$

which has the effect on a basis function of

$$n_1 \to n_1 - p_2, \quad n_2 \to n_2 + q_2, \quad A_1 \to A_1 + 2p_2.$$

and so

$$E = -2\omega(2n_0+2k_1 n_1+2k_2 n_2+\cdots) \to -2\omega(2n_0+2k_1(n_1-p_2)+2k_2(n_2+q_2)+\cdots)$$

$$= -2\omega(2n_0 + 2k_1 n_1 + 2k_2 n_2 + \cdots),$$

that is, E is unchanged. A similar lowering operator is

$$\Xi^-_{12} = \underbrace{K^{+\,A_1-2(p_2-1)}_{1\ n_1+(p_2-1)} \cdots K^{+\,A_1}_{1\ n_1}}_{p_2 \text{ terms}} \underbrace{J^-_{2\ n_2-(q_2-1)} \cdots J^-_{2\ n_2}}_{q_2 \text{ terms}}.$$

The $K^\pm$ operators differ from the 2D TTW procedure.

The transformation $n_1 \to -n_1 - A_1 - a_1 - 1$ while holding E constant has the effect of changing the sign of A_0.

Can check that

$$L_{01}^{+} = \Xi_{01}^{+} + \Xi_{01}^{-} \qquad \text{and} \qquad L_{01}^{-} = \frac{\Xi_{01}^{+} - \Xi_{01}^{-}}{A_0}$$

are polynomials in E, A_0^2 and A_1^2. Since

$$A_0^2 = k_1^2 - \ell_1 \quad \text{and} \quad A_1^2 = \frac{\frac{1}{4}k_2^2 - \ell_2}{k_1^2}$$

we can replace E, A_0^2 and A_1^2 with a second order differential operators where ever they appear in these expressions.

E.g. for $k_1, k_2, k_3 = 2, 1, 1$ these are operators that commute with H of orders 5 and 6 that are algebraically independent of the second order operators.

For the case $k_1, k_2, k_3 = 2, 1, 1$, L_{01}^{+} is a 5^{th} order operator.

$$L_{01}^{+} = \left(-\frac{2}{r^3}\partial_r + \frac{6}{r^4}\right) A_0^2 A_1^2 + \left(-\frac{\cos(4\theta_1)}{2r^3}\partial_r + \frac{\sin(4\theta_1)}{4r^4}\partial_{\theta_1} + \frac{1+5\cos(4\theta_1)}{2r^4}\right) {A_0}^4$$
$$+ \left(-\frac{1}{r}\partial_r + \frac{2}{r^2}\right) E {A_1}^2 + \left(\frac{\sin(4\theta_1)}{16}\partial_{\theta_1} + \frac{\cos(4\theta_1)}{4} + \frac{1}{8}\right) E^2$$
$$+ \left(-\frac{\cos(4\theta_1)}{4r}\partial_r + \frac{\sin(4\theta_1)}{4r^2}\partial_{\theta_1} + \frac{3\cos(4\theta_1)+1}{2r^2}\right) E {A_0}^2 + \left(-\frac{10}{r^3}\partial_r + \frac{4}{r^2}\partial_r^2 - \frac{6}{r^4}\right) {A_1}^2$$
$$- \left(\frac{\sin(4\theta_1)}{r}\partial_r\partial_{\theta_1} + \frac{3\cos(4\theta_1)+2-a_1^2}{r}\partial_r - \frac{5\sin(4\theta_1)}{4r^2}\partial_{\theta_1} - \frac{(6\cos(4\theta_1)+5-4a_1^2)}{2r^2}\right) E$$
$$+ \left(-\frac{\sin(4\theta_1)}{4r^2}\partial_r^2\partial_{\theta_1} + \frac{13\sin(4\theta_1)}{4r^3}\partial_r\partial_{\theta_1} - \frac{(4\cos(4\theta_1)+1)}{2r^2}\partial_r^2 + \frac{13+27\cos(4\theta_1)-4a_1^2}{2r^3}\partial_r \right.$$
$$\left. - \frac{5\sin(4\theta_1)}{r^4}\partial_{\theta_1} - \frac{25\cos(4\theta_1)+20-12a_1^2}{2r^4}\right) {A_0}^2$$
$$+ \frac{11\sin(4\theta_1)}{4r^2}\partial_r^2\partial_{\theta_1} + \frac{(11-8a_1^2+14\cos(4\theta_1))}{2r^2}\partial_r^2 - \frac{23\sin(4\theta_1)}{4r^3}\partial_r\partial_{\theta_1}$$
$$- \frac{26\cos(4\theta_1)+23-20a_1^2}{2r^3}\partial_r - \frac{21\sin(4\theta_1)}{4r^4}\partial_{\theta_1} - \frac{3(10\cos(4\theta_1)+7-4a_1^2)}{2r^4}$$

with the replacements $E \to H$, $A_0^2 \to k_1^2 - L_1$, $A_1^2 \to (k_2^2 - 4L_2)/(4k_1^2)$.

We find some polynomially closed subalgebras. For $i = 1, 2, 3$, define two differential operators polynomial in their arguments,

$$P_i^{(+)}(L_{i-1}, L_i, A_i^2) = \Xi_i^- \Xi_i^+ + \Xi_i^+ \Xi_i^- \quad \text{and} \quad P_i^{(-)}(L_{i-1}, L_i, A_i^2) = k_i \frac{\Xi_i^+ \Xi_i^- + \Xi_i^- \Xi_i^+}{A_{i-1}}.$$

For each for $i = 1, 2, 3$ we find,

$$\begin{aligned}
[L_i, L_i^-] &= -4k_i^2 q_i^2 L_i^- - 4\alpha_i k_i^2 q_i L_i^+ \\
[L_i, L_i^+] &= 2q_i\{L_i, L_i^-\} - 4k_i^2 q_i L_i^+ + 4k_i^2 q_i^2 L_i^- + 8q_i^3 k_i^2 L_i^- \\
[L_i^+, L_i^-] &= 2q_i(L_i^-)^2 - 2P_i^{(-)}(L_{i-1}, L_i, A_i^2)
\end{aligned}$$

and with $\alpha_1 = 1$, $\alpha_2 = 1/4$ and $\alpha_3 = 0$,

$$\{L_i, L_i^-, L_i^-\} + 2k_i^2(14q_i^2 - 3\alpha_i)(L_i^-)^2 + 6k_i^2(L_i^+)^2 + 6k_i^2 q_i\{L_i^+, L_i^-\} - 12k_i^2 P_i^{(+)} + 4k_i^2 q_i P_i^{(-)}$$

Then,

$$[L_j, L_i^\pm] = 0 \text{ for } i \neq j \quad \text{and} \quad [L_j^\pm, L_i^\pm] = [L_j^\pm, L_i^\mp] = 0 \text{ for } |i - j| > 1,$$

whereas $[L_i^\pm, L_{i+1}^\pm]$ and $[L_i^\pm, L_{i+1}^\mp]$ are related to the other symmetries by polynomial identities.

Conclusion

The TTW system can be extended to higher dimensions and leads to superintegrable systems in non-comformally-flat spaces.

One possible extension was presented here and required a conformally covariant deformation of the potential in order to use the raising and lowering operator method to prove superintegrability.

This suggests a 'natural' differential operator associated with a (conformal-) Killing tensor that commutes with a onformally covariant Laplacian other than the usual conformally covariant Laplacian $\nabla^2 - \mathcal{R}/6$.