Perimeter polynomials and scaling analysis for percolation problems

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Overview

1. Introduction to percolation
2. Basic scaling theory.
3. Numerical tests of scaling.
4. Amplitude estimates.

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The percolation probability $P(p)$ is the order parameter while the the average cluster size $S(p)$, which diverges as $p \to p_c^-$, plays a role similar to a susceptibility.

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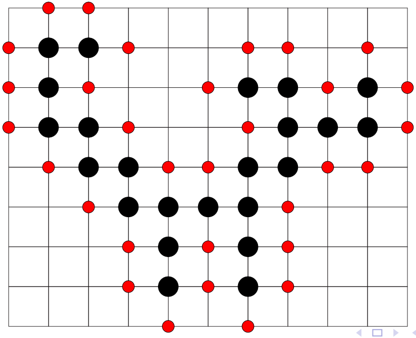

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Perimeter polynomials are defined as

$$D_s(q) = \sum_t g_{s,t} q^t$$

and from these we can find the average number of clusters of size s per vertex

$$n_s(p) = p^s D_s(1-p) = \sum_t g_{s,t} p^s (1-p)^t$$

Scaling theory

According to standard scaling theory the cluster numbers behave as

$$n_s(p) = s^{-\tau} f[(p - p_c)s^{\sigma}] \quad (p \to p_c,\ s \to \infty),$$

where $\tau = 187/91$ and $\sigma = 36/91$.

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$$\begin{aligned}
-P(p) &= \sum_s [n_s(p) - n_s(p_c)]s \quad \propto \int s^{1-\tau}[f(z) - f(0)]\mathrm{d}s \\
&= \frac{1}{\sigma}|p - p_c|^{(\tau-2)/\sigma} \int |z|^{-1+(2-\tau)/\sigma}[f(z) - f(0)]\mathrm{d}z \\
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The percolation probability $P(p)$ is zero for $p < p_c$ and it follows that

$$\int_{-\infty}^{0} |z|^{-1-\beta}[f(z) - f(0)]\mathrm{d}z = \int_{-\infty}^{0} |z|^{-\beta} \left[\frac{\mathrm{d}f}{\mathrm{d}z}\right] \mathrm{d}z = 0$$

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The corresponding integrals in the high-density region (that is the integral from 0 to ∞) should not vanish so as to give a non-zero percolation probability.

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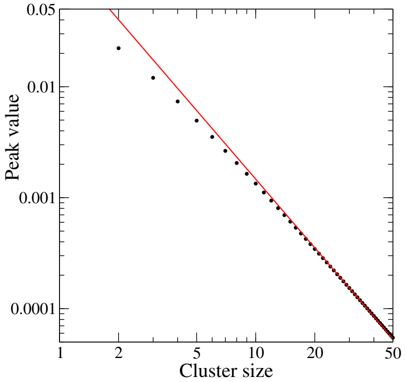

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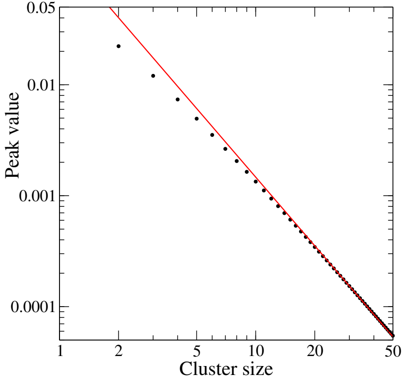

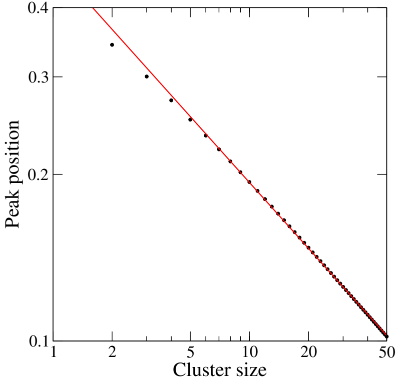

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$\bar{n}_s(z) = s^{\tau} n_s(z)$, with $z = (p - p_c)s^{\sigma}$, should converge to a unique distribution. Plots of $\bar{n}_s(z)$ vs. z for various values of s will exhibit a scaling collapse to a single curve.

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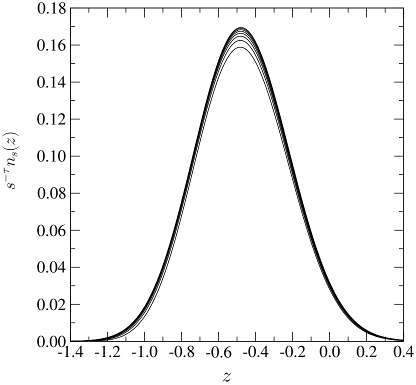

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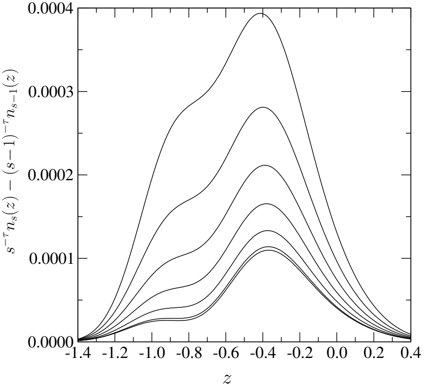

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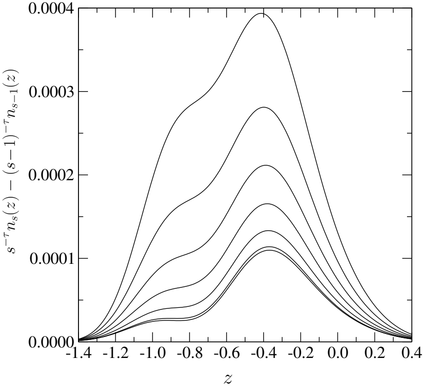

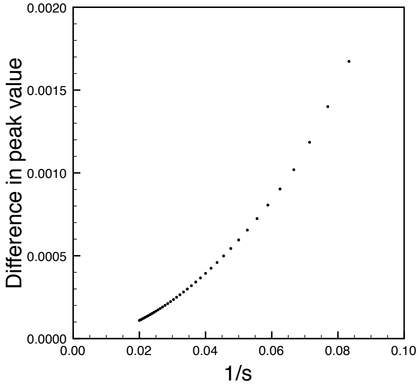

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Next we test whether the weighted integral of the derivative of the scaling function vanishes in the low-density region but is non-zero in the high-density region.

$$\int_{-\infty}^{0} |z|^{-1-\beta}[f(z) - f(0)]\mathrm{d}z = \int_{-\infty}^{0} |z|^{-\beta} \left[\frac{\mathrm{d}f}{\mathrm{d}z}\right] \mathrm{d}z = 0$$

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Integrals over the high-density region is over the interval $[0, s^\sigma(1 - p_c) = z_+]$.

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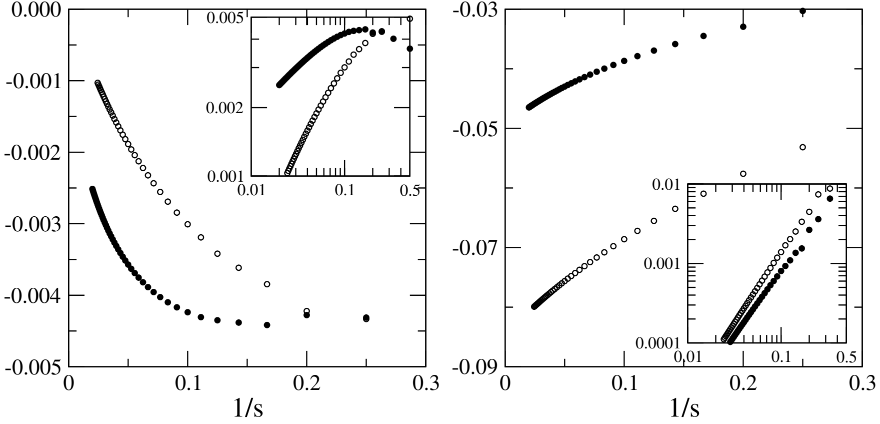

Asymmetry of scaled distribution

Start with the scaled distribution $\bar{n}_s(z) = s^\tau n_s(z)$, where $z = (p - p_c)s^\sigma$.
Let z_s denote the position of the maximum. Look at the asymmetry around z_s

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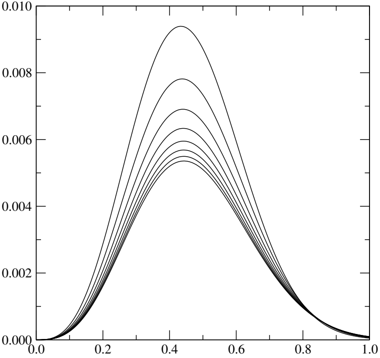

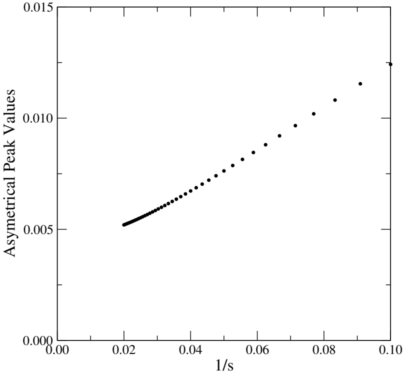

Scaling theory: Cluster Size

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S(p) &\propto \sum_s s^2 n_s(p) \quad \propto \int s^{2-\tau} f(z)\mathrm{d}s \\
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and in the low-density region

$$
I_s^- = \int_{-z_-}^{0} (-z)^{\gamma-1} \bar{n}_s(z)\mathrm{d}z = \int_0^{z_-} z^{\gamma-1} \sum_{k=0} a_k (-z)^k = \sum_{k=0} (-1)^k a_k (z_-)^{k+\gamma}/(k+\gamma),
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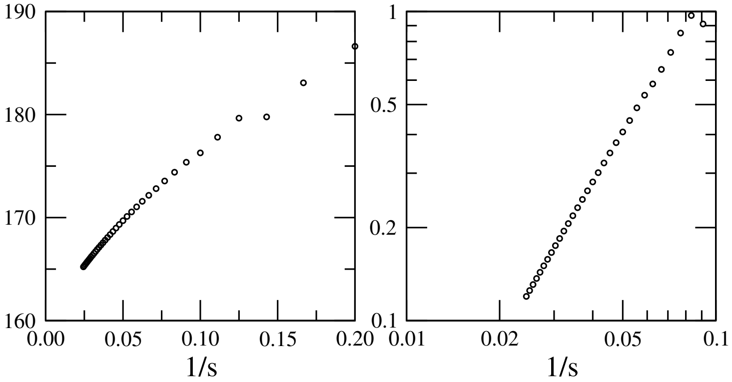
from which we obtain the estimate $\Gamma_s^-/\Gamma_s^+ = I_s^-/I_s^+$.

Amplitude ratios: Bond percolation



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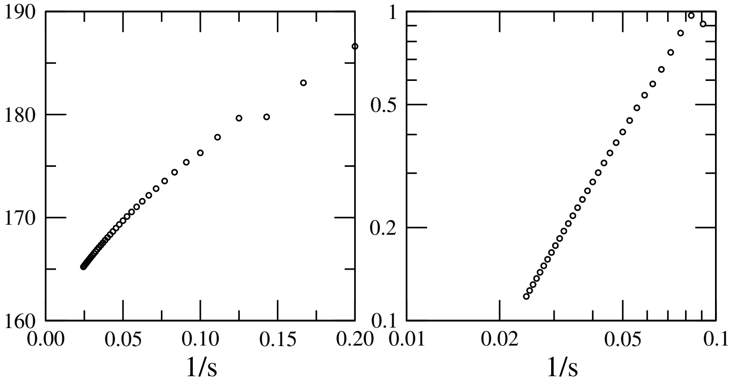

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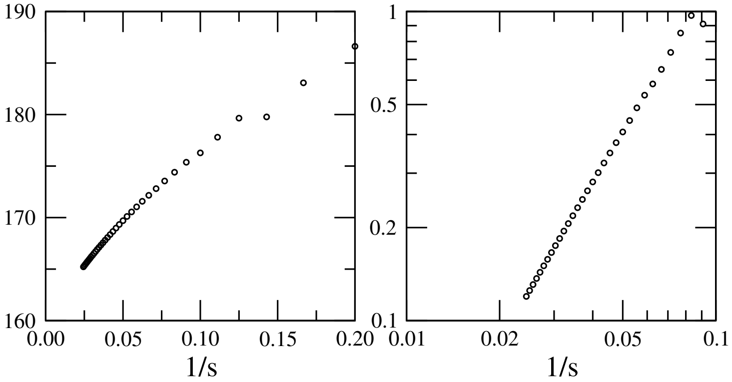

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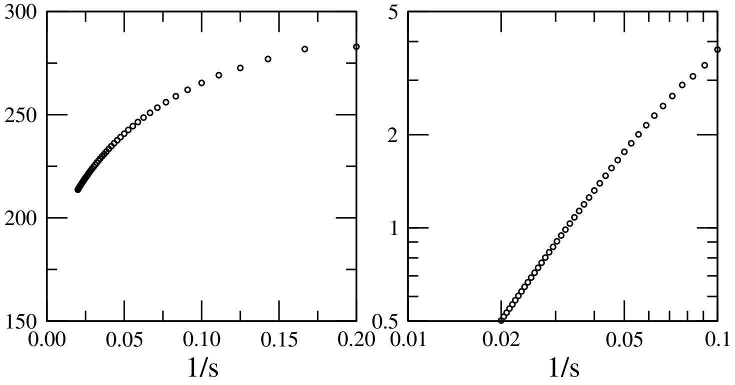
Extrapolation then give $\Gamma^-/\Gamma^+ = 159.2 \pm 0.2$.

Amplitude ratios: Site percolation



The amplitude ratios have pronounced curvature when plotted against $1/s$.

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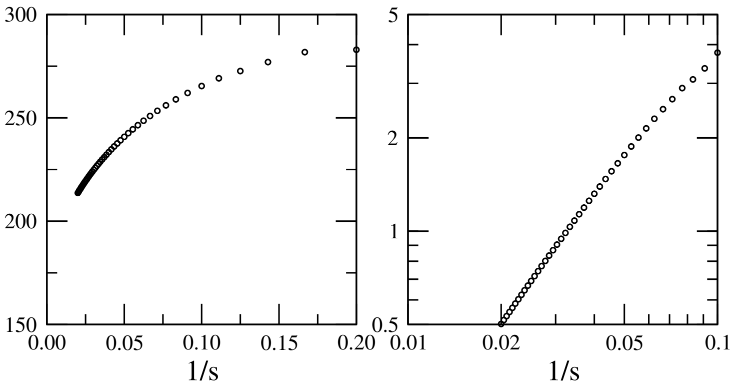

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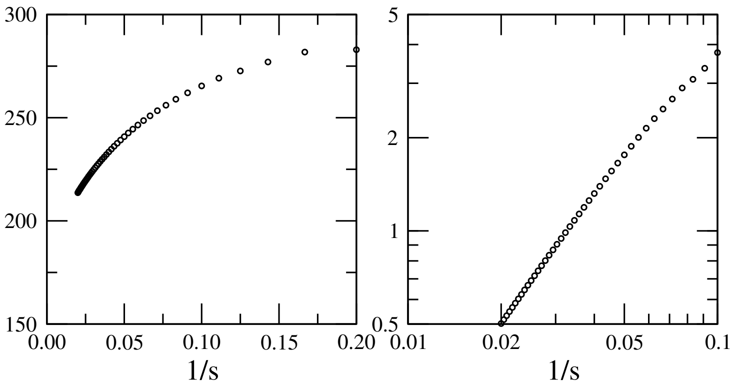

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A log-log plot of the difference between consecutive ratios clearly has a power-law decay with $1/s$. In this case the exponent work out to be around 0.55.

Extrapolation then give $\Gamma^{-}/\Gamma^{+} = 164.5 \pm 1.5$