# Algebraic structures from group character rings – an integrable model perspective

joint work with Berftried Fauser and Ronald C King


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# Some classical groups (and not-so-classical ones)

$$
\begin{aligned}
GL(N,\mathbb{C}) &= \{m_{i,j}, 1 \le i,j \le N : \det(m) \neq 0\},\\
U(N) &= \{m_{i,j}, 1 \le i,j \le N : \det(m) \neq 0,\ \&\ m^*_{ij} = m^{-1}_{ji}\},\\
O(N,\mathbb{C}) &= \{m_{i,j}, 1 \le i,j \le N : m^T m = I\},\\
Sp(N,\mathbb{C}) &= \{m_{i,j}, 1 \le i,j \le N : m^T J m = J, J = -J^T, \&\ \det(J) \neq 0\}.
\end{aligned}
$$

$$
\begin{aligned}
GL_1(N,\mathbb{C}) &= \{m_{i,j}, 1 \le i,j \le N : \det(m) \neq 0,\ \&\ \textstyle\sum_i m_{ij} = 1,\ \forall j\};\\
O_\eta(N,\mathbb{C}) &= \{m_{i,j}, 1 \le i,j \le N : m^T \eta\, m = \eta, \eta = \eta^T, \&\ \det(\eta) = 0\},\\
Sp(2K+1,\mathbb{C}) &= \{m_{i,j}, 1 \le i,j \le 2K+1 : m^T J m = J, J = -J^T\},
\end{aligned}
$$

$$
H_\pi(N,\mathbb{C}) = \{m_{i,j}, 1 \le i,j \le N : m \otimes \cdots \otimes m \circ T = T\}.
$$

# Group Characters 101

- The *character* $\chi(g)$ of a group $G$, evaluated on a group element $g$, in a representation $\pi$ is the trace of the matrix representing $g$, $\chi(g) = Tr(\pi(g))$. Let $g \in G$ let have eigenvalues $x_1, x_2, \cdots, x_N$. Then we have the following characters:

| | | |
|---|---|---|
| *defining* representation | $s_{(1)} = \sum_i x_i$ | $\boxed{i}$ |
| *symmetric* tensor rep | $s_{(2)} = \sum_{i \leq j} x_i x_j$ | $\boxed{i}\boxed{j}$ |
| *antisymmetric* tensor rep | $s_{(1,1)} = \sum_{i < j} x_i x_j$ | $\begin{array}{\|c\|}\hline i \\ \hline j \\ \hline\end{array}$ |

- **Theorems**
  (i)(Schur-Weyl) The characters of irreducible covariant tensor representations of $GL(N)$ are certain symmetric polynomials $s_\lambda(x_1, x_2, \cdots, x_N)$, defined as ratios of determinants, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is an integer partition visualised as a Ferrers diagram.
  (ii)The character formula can be written combinatorially via semistandard tableaux,

$$s_\lambda(x) = \sum_{T \in SST_\lambda} x^T$$

where $x^T$ is the monomial $x_1^{t_1} x_2^{t_2} \cdots x_N^{t_N}$, and the exponents are multiplicities of the respective entries in $T$ of shape $\lambda$.

# Universal character rings

- Let $X = \{x_1, x_2, x_3, \cdots\}$ be a denumerably infinite set of independent variables. For each partition $\lambda$ of finite weight $|\lambda| = d$, there is a "universal" Schur function $s_\lambda(X)$ such that each $s_\lambda(x_1, x_2, x_3, \cdots, x_N) \in \mathbb{C}[x_1, x_2, \cdots, x_N]_d^{\mathfrak{S}_N}$ is given by **specialisation**,

$$s_\lambda(x_1, x_2, x_3, \cdots, x_N) = s_\lambda(X), \qquad X \equiv (x_1, x_2, x_3, \cdots, x_N, 0, 0, \cdots).$$

- The ring generated by $s_\lambda(X)$ is denoted $\Lambda(X)$.
- **Theorems**

  (i) The ring Char-GL of universal characters of the general linear group is isomorphic to the ring $\Lambda(X)$ of symmetric functions in the alphabet $X$.

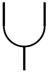
  (ii) The rings Char-O and Char-Sp of universal characters of the orthogonal and symplectic groups are isomorphic to Char-GL, but the basis and product laws differ.

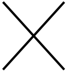
  (iii) *The ring* Char-$H_\pi$ *of universal characters of the* $H_\pi$ *subgroup of GL is likewise isomorphic to* Char-GL, *but these characters are generically decomposable rather than irreducible.*

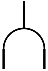
# Diagram calculus for Char-GL

 $\Leftrightarrow$ $\mathrm{Id}(s_\lambda) = s_\lambda$ (identity).

 $\Leftrightarrow$ $\eta(1) = s_0$ (unit).

 $\Leftrightarrow$ $\mathrm{m}(s_\lambda \otimes s_\mu) = s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$ (outer product).

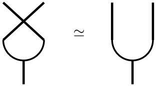
 $\Leftrightarrow$ $\mathrm{sw}(s_\mu \otimes s_\nu) = s_\nu \otimes s_\mu$ (switch).

 $\Leftrightarrow$ $\Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu \otimes s_\nu$ (outer coproduct).

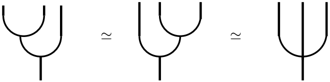
 $\Leftrightarrow$ $\varepsilon(s_\lambda) = \delta_{\lambda,(0)}$ (counit).

S  $\Leftrightarrow$ $\mathrm{S}(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$ (antipode).

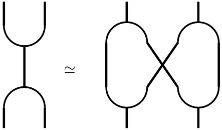
**Lemma**: The outer product and coproduct with $\eta$, $\varepsilon$, S form a **Hopf algebra** $(\Lambda, \mathsf{m}, \Delta, \eta, \varepsilon, \mathsf{S})$.



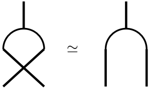
(commutativity).



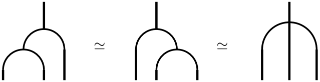
(associativity).



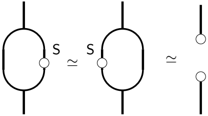
(bialgebra law $\Delta(fg) = \Delta(f)\Delta(g)$)



(cocommutativity).



(coassociativity).

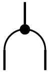


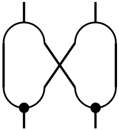

(antipode law).

**Inner product $*$:**

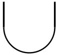
Via the Frobenius map $\varphi$ : Char-GL $\to$ Char-$\mathfrak{S}$, $s_\lambda \mapsto \chi^\lambda$, define the **inner product** $f * g = \varphi^{-1}\big(\varphi(f) \cdot \varphi(g)\big)$.

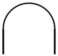
 $\Leftrightarrow$ $m(s_\lambda \otimes s_\mu) = s_\lambda * s_\mu = \sum_\nu g^\nu_{\lambda,\mu} s_\nu$ (inner product).

 $\Leftrightarrow$ $\delta(s_\lambda) = \sum_{\mu,\nu} g^\lambda_{\mu,\nu} s_\mu \otimes s_\nu$ (inner coproduct).

 $\simeq$  $\Leftrightarrow$ $\Delta(f * g) = \Delta(f) * \Delta(g)$ (mixed bialgebra)

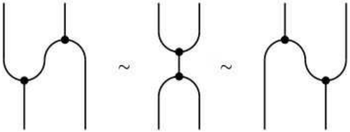
 $\Leftrightarrow$ $1_m = M(X) = \prod_i (1 - x_i)^{-1}$ (inner unit).

 $\Leftrightarrow$ $\langle s_\mu \mid s_\nu \rangle := \varepsilon(s_\mu * s_\nu) \equiv \delta_{\mu,\nu}$ (Schur-Hall scalar p).

 $\Leftrightarrow$ $\delta(M) = \sum_\lambda s_\lambda \otimes s_\lambda \equiv M(XY)$ (Cauchy kernel).

**Lemma**:
The inner (co)product forms a **Frobenius algebra** $(\Lambda, *, \delta, 1_m = M)$ with compatibility

$$(f \otimes 1_m) * \delta(g) = \delta(f * g) = \delta(f) * (1_m \otimes g)$$



Heinz Hopf, 1894-1971

Ferdinand Georg Frobenius, 1817-1849

# Symmetric function series

- To handle universal characters of subgroups we need to define series, viz. $Z = \sum_{\zeta \in Z} s_\zeta t^{|\zeta|}$ (working formally in $\Lambda[[t]]$).
- Examples

$$M := \prod_i \frac{1}{(1-x_i)} = 1 + \sum_i x_i + \sum_{i \leq j} x_i x_j + \sum_{i \leq j \leq k} x_i x_j x_k + \cdots = \sum_0^\infty s_{(n)}$$

$$L := \prod_i (1-x_i) = 1 - \sum_i x_i + \sum_{i<j} x_i x_j - \sum_{i<j<k} x_i x_j x_k + \cdots = \sum_0^\infty (-1)^n s_{(1^n)}$$

$$M_{(2)} := \prod_{i \leq j} \frac{1}{(1-x_i x_j)} = D \qquad L_{(2)} := \prod_{i \leq j} (1 - x_i x_j) = C$$

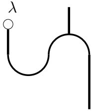
$$M_{(1,1)} := \prod_{i<j} \frac{1}{(1-x_i x_j)} = B \qquad L_{(1,1)} := \prod_{i<j} (1 - x_i x_j) = A$$

$$M_\pi := \prod_{T \in SST_\pi} \frac{1}{(1-x^T)} \qquad L_\pi := \prod_{T \in SST_\pi} (1 - x^T)$$

- The derived series are defined via symmetric function composition or *plethysm*, $f[g](X)$.

# Skew

- Skew is the adjoint of outer multiplication,

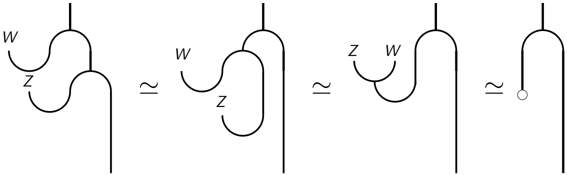



$$\Leftrightarrow \quad s_\lambda^{\perp}(s_\mu) = s_{\lambda/\mu} = \sum \langle s_\lambda | s_\mu^{(1)} \rangle s_\mu^{(2)} \qquad \text{(skew by } s_\lambda\text{)}.$$

- For a series $Z = \sum_{\zeta \in Z} t^{|\zeta|} s_\zeta$ extend linearly,

$$s_{\lambda/Z} = \sum_{\zeta \in Z} s_{\lambda/\zeta}$$

- If $W$ and $Z$ are inverses $WZ = 1$, we have $\big(s_\lambda/W\big)/Z = s_\lambda/(WZ) \equiv s_\lambda$:



# GL subgroups – characters, branching rules, products

- The rings Char-O , Char-Sp of universal characters for the orthogonal and symplectic groups are isomorphic to Char-GL via skewing by series, $/L_{(2)} \equiv /C$ and $/L_{(1,1)} \equiv /A$.
- Littlewood denoted universal general linear, orthogonal and symplectic characters by $\{\lambda\}$, $[\lambda]$, $\langle\lambda\rangle$ (with alphabet $X$ understood):

$$[\lambda] = \sum_{\gamma \in C} \{\lambda/\gamma\}; \qquad \langle\lambda\rangle = \sum_{\alpha \in A} \{\lambda/\alpha\};$$

- *Similarly, symmetric functions* $((\lambda))$ *of* $H_\pi$ *type are defined by the isomorphism* $/M_\pi$ : Char-GL $\rightarrow$ *Char-*$H_\pi$*:*

$$((\lambda)) = \sum_{\mu \in L_\pi} \{\lambda/\mu\};$$

- The **inverse** isomorphisms $/D = /C^{-1}$, $/B = /A^{-1}$, $/L_\pi = /M_\pi^{-1}$, correspond to **group branching**, and give the rules for resolving GL representations into a sum of representations of the respective subgroups (indecomposable in the case of $H_\pi$).

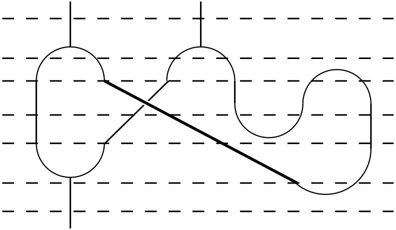
## Newell-Littlewood rule

$$[\lambda][\mu] = \sum_{\alpha} [\lambda/\alpha \cdot \mu/\alpha], \qquad \langle\lambda\rangle\langle\mu\rangle = \sum_{\alpha} \langle\lambda/\alpha \cdot \mu/\alpha\rangle.$$

$\pi$-**Newell Littlewood rule** (Fauser, PDJ, King, Wybourne 2006)

$$((\lambda))((\mu)) = \sum_{\alpha_k} ((\lambda/ \prod_{k=1}^{p} \alpha_k[\pi'_{(1)}] \cdot \mu/ \prod_{k=1}^{p} \alpha_k[\pi'_{(2)}]))$$

Dudley E Littlewood, 1903-1979



$$\Leftrightarrow \quad \begin{array}{l} \lambda \otimes \mu \\ \lambda_{(1)} \otimes \lambda_{(2)} \otimes \mu_{(1)} \otimes \mu_{(2)} \\ \sum_\alpha \lambda_{(1)} \otimes \lambda_{(2)} \otimes \mu_{(1)} \otimes \mu_{(2)} \otimes \alpha \otimes \alpha \\ \sum_\alpha \lambda_{(1)} \otimes \mu_{(1)} \otimes \lambda_{(2)} \otimes \mu_{(2)} \otimes \alpha \otimes \alpha \\ \sum_\alpha \lambda_{(1)} \otimes \mu_{(1)} \otimes \lambda_{(2)} \otimes \langle \mu_{(2)} | \alpha \rangle \alpha \\ \sum_\alpha \lambda_{(1)} \cdot \mu_{(1)} \otimes \lambda_{(2)} \otimes \langle \mu_{(2)} | \alpha \rangle \alpha \\ \sum_\alpha \lambda_{(1)} \cdot \mu_{(1)} \langle \lambda_{(2)} | \alpha \rangle \langle \mu_{(2)} | \alpha \rangle \end{array} \quad .$$

# Explicit forms for subgroup characters

- **Generating functions:**
  For the following we adopt the notation $s_\lambda^{(\pi)}(X)$ for the universal Char-$\mathrm{H}_\pi$characters (omitting the superfix for ordinary $S$-functions).
  **Lemma** (Fauser, PDJ, King 2010)

$$L_\pi(Z)M(XZ) = \sum_\lambda s_\lambda^{(\pi)}(X)s_\lambda(Z)$$

- Examples

$$\prod_{i\leq j}(1-z_iz_j)\prod_{i,j}(1-x_iz_j)^{-1} = \sum_\lambda s_\lambda^{(2)}(X)s_\lambda(Z); \qquad \text{(orthogonal)}$$

$$\prod_{i<j}(1-z_iz_j)\prod_{i,j}(1-x_iz_j)^{-1} = \sum_\lambda s_\lambda^{(1^2)}(X)s_\lambda(Z); \qquad \text{(symplectic)}$$

$$\prod_{i\leq j\leq k}(1-z_iz_jz_k)\prod_{i,j}(1-x_iz_j)^{-1} = \sum_\lambda s_\lambda^{(3)}(X)s_\lambda(Z);$$

$$\prod_{i\neq j}(1-z_i^2z_j)\prod_{i<j<k}(1-z_iz_jz_k)^2\prod_{i,j}(1-x_iz_j)^{-1} = \sum_\lambda s_\lambda^{(21)}(X)s_\lambda(Z);$$

$$\prod_{i<j<k}(1-z_iz_jz_k)\prod_{i,j}(1-x_iz_j)^{-1} = \sum_\lambda s_\lambda^{(1^3)}(X)s_\lambda(Z).$$

# Vertex operator realisations

For the following we work in the extension $\Lambda[[z]]$ keeping the degree counting parameters $z_1, z_2, \cdots$ and suppressing the underlying alphabet $X$. Thus $M(z,X) = \prod_i (1 - zx_i)^{-1} = \sum z^n s_{(n)}$ is written just as $M(z)$.

- ***S*-functions $s_\lambda$:**
  The standard (Bernstein) form of vertex operator is $V(z) := M(z)L^\perp(1/z)$. Let $Z = (z_1, z_2, \cdots, z_\ell)$ and for any partition $\lambda$ whose number of parts is $\leq \ell$ let $Z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_\ell^{\lambda_\ell}$ and $[Z^\lambda](\,\cdot\,)$ be the coefficient of $Z^\lambda$ in $(\,\cdot\,)$. Then

$$s_\lambda(z_1, z_2, \cdots, z_\ell) = [Z^\lambda] V(z_1) V(z_2) \cdots V(z_\ell) \cdot 1\,.$$

- **Symmetric functions $s_\lambda^{(\pi)}$** (Baker 1995; Fauser, PDJ, King 2010):
  Let $\pi$ be a partition of weight $p \geq 1$. Then

$$s_\lambda^{(\pi)}(z_1, z_2, \cdots, z_\ell) = [Z^\lambda] V^\pi(z_1) V^\pi(z_2) \cdots V^\pi(z_\ell) \cdot 1\,.$$

  with

$$V^\pi(z) = (1 - z^p \delta_{\pi,(p)}) M(z) L^\perp(1/z) \prod_{k=1}^{p-1} L^\perp_{\pi/(k)}(z^k)$$

# Vertex operators -ctd

Vertex operators belong to $End(\Lambda)$, and can be written working in the algebraic basis of power sum functions $p_k(X) = \sum x_i^k$ and realising the duals $p_k^\perp$ as $k\partial/\partial p_k$ (equivalent to a Heisenberg algebra). From the series expansion for $\ln(1-x)$ it follows that $M(z) = \exp\left(\sum_{k\geq 1} z^k p_k/k\right)$ and thus

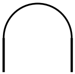
$$V(z) = (1-z)\exp\left(\sum_{k\geq 1}\frac{z^k}{k}p_k\right)\exp\left(-\sum_{k\geq 1} z^{-k}\frac{\partial}{\partial p_k}\right);$$

Similar mode sum expressions can be worked out for the universal subgroup characters; for example

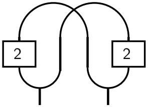
$$V^{(21)}(z) = (1-z)\exp\left(\sum_{k\geq 1}\frac{z^k}{k}p_k\right)\exp\left(-\sum_{k\geq 1}(z^{-k}+z^k)\frac{\partial}{\partial p_k} + kz^k\frac{\partial^2}{\partial p_k^2}\right).$$

# The Cauchy kernel revisited

- The subgroup products are deformations via convolution with a 0-2 tangle (co-scalar product) which in the case of Char-O and Char-Sp is identically the Cauchy kernel, i.e. $r_{(2)} = r_{(1,1)} = M(XY)$
- The general 0-2 tangle $r_\pi$ is a convolutive product of $p = |\Delta'\pi|$ Cauchy kernels, whose downward lines are modified by the insertion of plethysms coming from the corresponding *cut coproduct* parts of $\pi$:

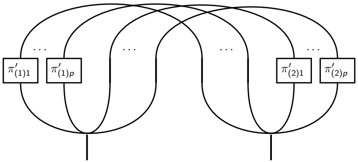


$$\Leftrightarrow \qquad r_{(2)} = r_{(1,1)} \;::\; 1 \mapsto \sum\nolimits_{\alpha} \alpha \otimes \alpha,$$




$$\Leftrightarrow \qquad r_{(3)} \;::\; 1 \mapsto \sum\nolimits_{\alpha,\beta} \alpha[2]\cdot\beta \otimes \alpha\cdot\beta[2],$$

$$r_\pi \quad \cong$$

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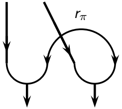
# Main results

- Associated with the 0-2 tangle $r_\pi$ we define the 2-2 tangle $R_\pi$,

$R_\pi \cong$ 


$$R_\pi(f \otimes g) = \sum_\alpha f \cdot \prod_{k=1}^{p} \alpha_k[\pi'_{(1)}] \otimes g \cdot \prod_{k=1}^{p} \alpha_k[\pi'_{(2)}].$$

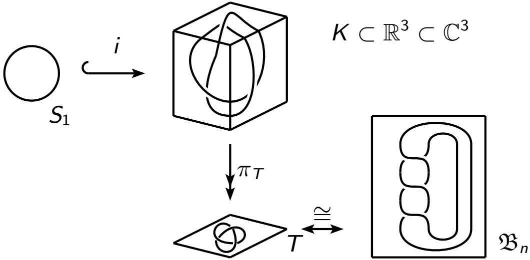
- **Theorem**
(i) The co-scalar product $r_\pi$ is a co-quasitriangular structure on the outer Hopf algebra $\Lambda$.
(ii) Dually, $\Lambda$ with $R_\pi$ as defined above is a braided Hopf algebra. $R_\pi$ satisfies the Yang-Baxter relation

$$R_\pi^{12} R_\pi^{13} R_\pi^{23} = R_\pi^{23} R_\pi^{13} R_\pi^{12}.$$

and the object $\mathrm{c}^\pi := \mathrm{sw} \circ R_\pi$ is a braid.

(**Proof**: the central fact is that the dual of $r_\pi$ is a 2-cocycle in the appropriate cohomology; other structural axioms come from the properties of outer product, antipode and other elements as already considered).

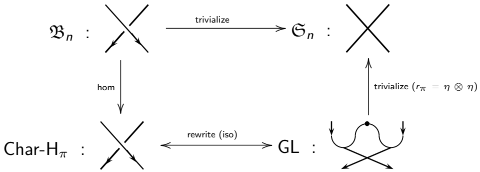
# Knot projections and the braid alphabet




braid-alphabet $=$ $\{$  $|$ relations $\}$

# Char-$H_\pi$ as a ribbon Hopf algebra

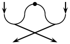
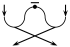
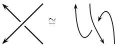
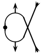
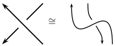
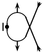
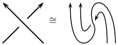
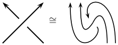
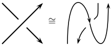
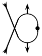
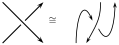
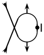
- We want to get information on the braid group by working with diagram calculus in Char-$H_\pi$ according to the scheme:

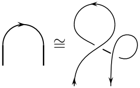
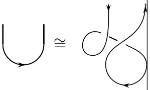
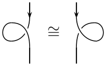
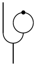
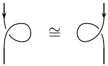
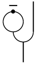



- The diagrammatic rules do not permit removal of line twists, or writhes, so we adopt the modified Reidemeister $\mathbf{R1}'$ move
- Adopt the simplified notation $r_\pi = \sum \alpha^\pi_{(1)} \otimes \alpha^\pi_{(2)}$ to symbolise the running variables in the Sweedler sums, and also define $Q_\pi := \prod \alpha^\pi_{(1)} \cdot \alpha^\pi_{(2)}$

**Lemma**

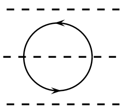
For each integer partition $\pi$ the universal character ring Char-$H_\pi$, with outer Hopf algebra inherited from Char-GL , is a ribbon Hopf algebra with writhe element $Q_\pi$ and braid alphabet given by the following dictionary $\cdots$

| braid | Char-$H_\pi$ tangle | Char-GL tangle | algebraic expression |
|---|---|---|---|
| $c^{\pi}_{\Lambda\Lambda}$ |  |  | $\lambda \otimes \mu \quad \mapsto \quad \sum_\alpha \mu \cdot \alpha^\pi_{(2)} \otimes \lambda \cdot \alpha^\pi_{(1)}$ |
| $\overline{c}^{\pi}_{\Lambda\Lambda}$ |  |  | $\lambda \otimes \mu \quad \mapsto \quad \sum_\alpha \mu \cdot \mathsf{S}(\alpha^\pi_{(2)}) \otimes \lambda \cdot \alpha^\pi_{(1)}$ |
| $c^{\pi}_{\Lambda^*\Lambda}$ |  |  | $\lambda^* \otimes \mu \quad \mapsto \quad \sum_\alpha \mu \cdot \alpha^\pi_{(2)} \otimes (\lambda/\alpha^\pi_{(1)})^*$ |
| $\overline{c}^{\pi}_{\Lambda^*\Lambda}$ |  |  | $\lambda^* \otimes \mu \quad \mapsto \quad \sum_\alpha \mu \cdot \mathsf{S}(\alpha^\pi_{(2)}) \otimes (\lambda/\alpha^\pi_{(1)})^*$ |
| $c^{\pi}_{\Lambda^*\Lambda^*}$ |  |  | $\lambda^* \otimes \mu^* \quad \mapsto \quad \sum_\alpha (\mu/\alpha^\pi_{(2)})^* \otimes (\lambda/\alpha^\pi_{(1)})^*$ |
| $\overline{c}^{\pi}_{\Lambda^*\Lambda^*}$ |  |  | $\lambda^* \otimes \mu^* \quad \mapsto \quad \sum_\alpha (\mu/\mathsf{S}(\alpha^\pi_{(2)}))^* \otimes (\lambda/\alpha^\pi_{(1)})^*$ |
| $c^{\pi}_{\Lambda\Lambda^*}$ |  |  | $\lambda \otimes \mu^* \quad \mapsto \quad \sum_\alpha (\mu/\alpha^\pi_{(2)})^* \otimes \lambda \cdot \alpha^\pi_{(1)}$ |
| $\overline{c}^{\pi}_{\Lambda\Lambda^*}$ |  |  | $\lambda \otimes \mu^* \quad \mapsto \quad \sum_\alpha (\mu/\mathsf{S}(\alpha^\pi_{(2)}))^* \otimes \lambda \cdot \alpha^\pi_{(1)}$ |

| map | Char-$H_\pi$ tangle | Char-GL tangle | algebraic expression |
|---|---|---|---|
| b |  |  | $1 \mapsto \sum \sigma \otimes \sigma^*$ |
| d |  |  | $\lambda^* \otimes \mu \mapsto \langle \lambda \vert \mu \rangle$ |
| $\overline{\mathtt{b}}_\pi$ |  |  | $1 \mapsto \sum \rho^* \otimes \rho$ |
| $\overline{\mathtt{d}}_\pi$ |  |  | $\lambda \otimes \mu^* \mapsto \langle \mu \vert \lambda \rangle$ |
| $\theta_\pi$ |  |  | $\lambda \mapsto Q_\pi \cdot \lambda$ |
| $(\theta_\pi)^{-1}$ |  |  | $\lambda \mapsto (Q_\pi)^{-1} \cdot \lambda$ |

# Knot invariant operators

- Complete knots and links are projected as decorated images of products of circles, and so must be interpreted in terms of slicings of 0-0 tangles. Consider

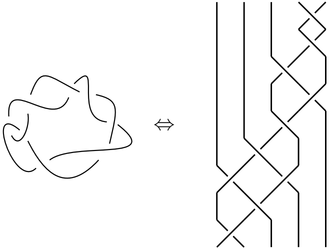


$$\Leftrightarrow \qquad \begin{array}{rcl} 1 & \mapsto & \\ & \mapsto & \sum_\sigma \sigma \otimes \sigma^* \\ & \mapsto & \sum_\sigma \langle \sigma | \sigma \rangle \equiv \sum_\sigma 1 = \infty \end{array}$$

- Instead we cut and open one strand (or more) of the 0-0 tangle, and evaluate the resulting 1-1 tangle invariant as an element of $End(\Lambda)$ (or of $End(\otimes^k \Lambda)$ for a $k$-$k$ tangle).
- A knot $K$ is isotopic to the closure of a braid element in $\mathfrak{B}_m$, by means of a word of length $\ell$, $b_K = b_{i_1}{}^{e_1} b_{i_2}{}^{e_2} \cdots b_{i_\ell}{}^{e_\ell}$, where each $i_k \in \{1, 2, \cdots, m-1\}$, with each exponent $e_i = \pm 1$.
- We have $\ell \equiv \sum_i |e_i|$, while the sum $w_K = \sum_i e_i := w_+ - w_-$ is the writhe of the knot or link projection (the difference between the positive and negative exponent sums).

# Knot invariant operators -ctd

- For example for the knot 8_1 and its braid representation we have



- The image of $b_{8\_1}$ under the homomorphism $\mathfrak{B}_5 \to \mathfrak{S}_5$ is the 5-cycle (42531). Labelling the 5 downward braid strands with $S$-functions $\sigma_1$ (open), $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$, after braiding but before closing cups, the element is

$$\sum \sigma_4 \cdot (\alpha_1\alpha_2\alpha_7\alpha_9\alpha_{10}) \otimes \sigma_5 \cdot (\alpha_1\alpha_2\alpha_4\alpha_5\alpha_6\alpha_8) \otimes$$
$$\otimes\, \sigma_1 \cdot (\alpha_8\alpha_9) \otimes \sigma_2 \cdot (\alpha_6\alpha_7) \otimes \sigma_3 \cdot (\alpha_3\alpha_4) \otimes \sigma_2 \otimes \cdots \otimes \sigma_5,$$

with the $\alpha \cdots \alpha$ standing for the summations over Sweedler part plethysms associated with the 10 crossings in this case.

# Knot invariant operators -ctd

- Closing the braids enforces identification between respective $S$-functions and (*allowing for antipodes*) restores the distributed parts to a product of ${Q_\pi}^{\pm 1}$ factors so that the 8_1 knot invariant operator (on $\sigma_1$) in this case is $\sigma_1 \to (Q_\pi)^{-2}\sigma_1$.

- **Theorem** (Fauser, PDJ, King 2012)
  (i) The $H_\pi$ invariant in $End(\Lambda)$ for a knot $K$ is

$$\mathcal{I}_K = (Q_\pi)^{w_K}.$$

  (ii) The $H_\pi$ invariant in $End(\Lambda \otimes \Lambda)$ for a 2-component link $L$ (with braid presentation cut on each component knot $K_1$, $K_2$) is

$$\mathcal{I}_L = (Q_\pi)^{w_1} \otimes (Q_\pi)^{w_2} \cdot (r_\pi)^{w_{12}}$$

  where $w_{12}$ is the *linking number* of the two knots.

B. Fauser and P. D. Jarvis.
A Hopf laboratory for symmetric functions.
*Journal of Physics A: Mathematical & General*, 37:1633–1663, February 2004.

B. Fauser, P. D. Jarvis, R. C. King, and B. G. Wybourne.
New branching rules induced by plethysm.
*Journal of Physics A: Mathematical &General*, 39:2611–2655, March 2006.

B. Fauser, P. D. Jarvis and R. C. King,
Plethysms, replicated Schur functions and series, with applications to vertex operators.
*Journal of Physics A: Theoretical & General* 43:405201, 2010

B. Fauser, P. D. Jarvis and R. C. King,
Hopf algebra structure of the character rings of orthogonal and symplectic groups.
*Journal of Physics A: Theoretical & General*, 2012 (accepted)

B. Fauser, P. D. Jarvis and R. C. King,
Ribbon Hopf algebras from group character rings.
*submitted*, 2012

B. Fauser, P. D. Jarvis and R. C. King,
Laplace pairings and hash products: a unified approach to the decomposition of group characters.
*in preparation*, 2012


... Brian realized that this is an elementary example of what Littlewood has called a *plethysm*, which treats the symmetry of the products of objects that themselves possess symmetry. Elliott had used plethysms in his nuclear studies, but no one had noticed their relevance to atomic shell theory before. At a conference at the US National Bureau of Standards in 1967, Brian unflinchingly described the details of the mathematics. The audience was stunned. At the end of Brians presentation a despairing voice asked, 'What *is* a plethysm?' We were all surprised to hear Brian say that a full explanation would take too much time...
– B R Judd, **Interaction with Brian Wybourne**, 2004

**Brian Garner Wybourne (1935–2003)**

`Schur`: *An Interactive Program For Calculating Properties Of Lie Groups and Symmetric Functions*, `http://sourceforge.net/projects/schur`

# Example: $s_{(1^2)}[s_{(2)}]$ in $GL(3)$

- The semi-standard tableaux $T$ (and monomials $x^T$) for $\begin{array}{|c|c|}\hline \phantom{1}&\phantom{1}\\\hline\end{array}$ are

$$SST_{\begin{array}{|c|c|}\hline \phantom{1}&\phantom{1}\\\hline\end{array}} = \left\{ \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&3\\\hline\end{array} \right\},$$

$$\therefore\ s_{(2)} = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

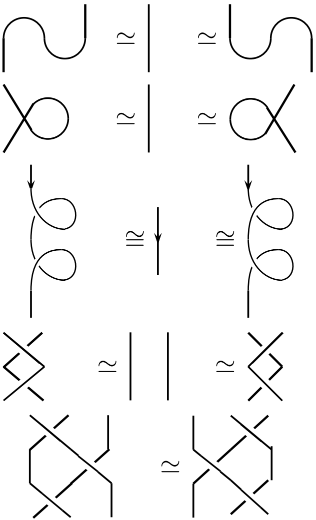
– that is, the alphabet $Y = \{x^T\} \equiv (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)$.

- Forming $\sum_{T<T'} x^T x^{T'}$ gives 15 monomials,

$$x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 +$$
$$x_1^2x_2x_3 + x_1x_2^3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1x_2^2x_3 +$$
$$x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 + x_2^2x_3^2 + x_2x_3^3$$

$$\therefore\ s_{(1^2)}[s_{(2)}] = s_{(3,1)} \leftrightarrow \left\{ \begin{array}{l} \begin{array}{ccc}1&1&1\\2\end{array}, \begin{array}{ccc}1&1&1\\3\end{array}, \begin{array}{ccc}1&1&2\\2\end{array}, \begin{array}{ccc}1&1&2\\3\end{array}, \begin{array}{ccc}1&1&3\\3\end{array}, \\ \begin{array}{ccc}1&1&3\\2\end{array}, \begin{array}{ccc}1&2&2\\2\end{array}, \begin{array}{ccc}1&2&2\\3\end{array}, \begin{array}{ccc}1&2&3\\3\end{array}, \begin{array}{ccc}1&2&3\\2\end{array}, \\ \begin{array}{ccc}1&3&3\\2\end{array}, \begin{array}{ccc}1&3&3\\3\end{array}, \begin{array}{ccc}2&2&2\\3\end{array}, \begin{array}{ccc}2&2&3\\3\end{array}, \begin{array}{ccc}2&3&3\\3\end{array}. \end{array} \right\}.$$

# Relations in the braid group $\mathfrak{B}_n$



**R0:** The zeroth or topological Reidemeister move ( $\cong$ closure).

**R1:** The first Reidemeister move.

**R1':** The first' Reidemeister move.

**R2:** The second Reidemeister move.

**R3:** The third Reidemeister move, ( $\cong$ Yang-Baxter equation).