A multi-dimensional ${}_1\psi_1$ sum and some related topics

(joint work with P.J. Forrester)

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Fix $0 < q < 1$ and define $(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$, $(a;q)_N := \dfrac{(a;q)_\infty}{(aq^N;q)_\infty}$.

Ramanujan's ${}_1\psi_1$ summation formula

$$\sum_{\nu=-\infty}^{\infty} \frac{(a;q)_\nu}{(b;q)_\nu} x^\nu = \frac{(ax;q)_\infty (q;q)_\infty (b/a;q)_\infty (q/ax;q)_\infty}{(x;q)_\infty (b;q)_\infty (q/a;q)_\infty (b/ax;q)_\infty},$$

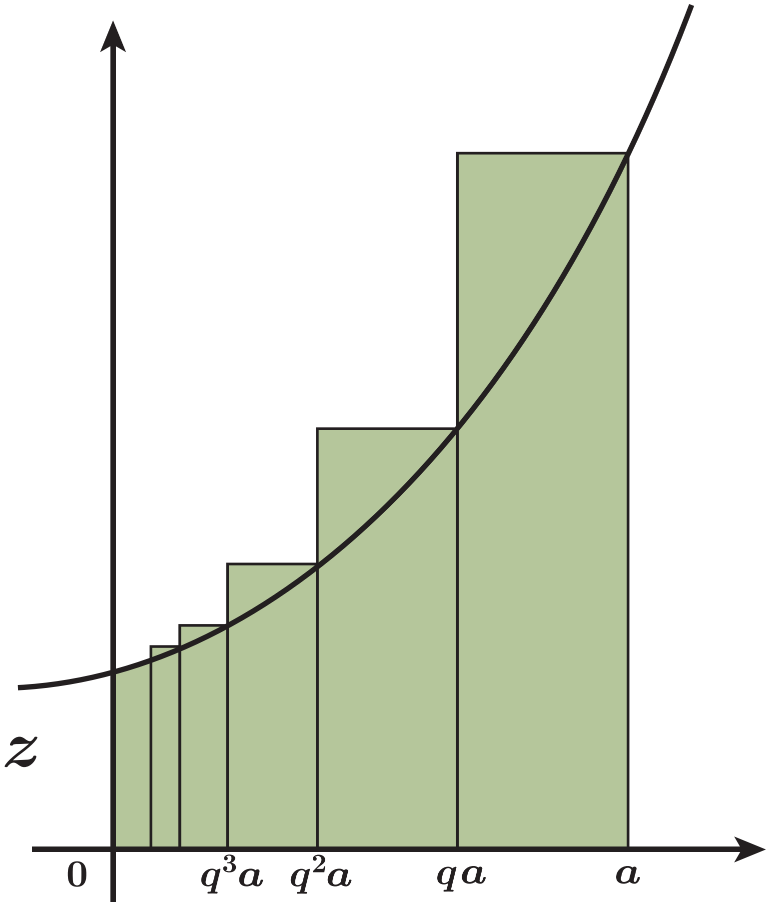
where $|b/a| < |x| < 1$.

As a special case, if $b = q$, then we have the q-binomial theorem

$$\sum_{\nu=0}^{\infty} \frac{(a;q)_\nu}{(q;q)_\nu} x^\nu = \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

This formula can be rewritten as the relation between the beta function and the gamma function via the q-integral representation.

q-Integral (Jackson integral)



$$\int_0^a f(z)d_qz := (1-q)\sum_{\nu=0}^{\infty} f(aq^\nu)aq^\nu$$

$$\xrightarrow[q\to 1]{} \int_0^a f(z)dz$$

$$\int_a^b f(z)d_qz := \int_0^b f(z)d_qz - \int_0^a f(z)d_qz$$

q-Beta function

$$B_q(\alpha,\beta) := \int_0^1 z^\alpha \frac{(qz;q)_\infty}{(q^\beta z;q)_\infty}\frac{d_qz}{z} \xrightarrow[q\to 1]{} \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz$$

where the Jackson integral of positive measure is given by

$$\int_0^a f(z)\frac{d_qz}{z} := (1-q)\sum_{\nu=0}^{\infty} f(aq^\nu).$$

q-Gamma function

$$\Gamma_q(\alpha) := \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha} \quad \xrightarrow[q\to 1]{} \quad \Gamma(\alpha)$$

q-Beta function and q-Gamma function

$$\boxed{B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}}$$

By definition

$$\mathbf{LHS} = (1-q)\sum_{\nu=0}^{\infty} q^{\nu\alpha}\frac{(q^{1+\nu};q)_\infty}{(q^{\beta+\nu};q)_\infty} \quad \mathbf{RHS} = (1-q)\frac{(q^{\alpha+\beta};q)_\infty (q;q)_\infty}{(q^\alpha;q)_\infty (q^\beta;q)_\infty}$$

This is equivalent to the q-binomial theorem!

$$\sum_{\nu=0}^{\infty}\frac{(a;q)_\nu}{(q;q)_\nu}x^\nu = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

Extended (bilateral) Jackson integral

Moreover, we extend the definition of the Jackson integral as follows, if it converges.

$$\int_0^{x\infty} f(z)\frac{d_qz}{z} := (1-q)\sum_{\nu=-\infty}^{\infty} f(xq^\nu).$$

By definition the extended Jackson integral is invariant under the q-shift $x \to qx$.

Extended q-beta integral

$$B_q(\alpha,\beta;x) := \int_0^{x\infty} z^\alpha \frac{(qz;q)_\infty}{(q^\beta z;q)_\infty}\frac{d_qz}{z}.$$

If $x=1$, then $B_q(\alpha,\beta;1)=B_q(\alpha,\beta)$, because the integrand vanishes when $z=q^{-1},q^{-2},\ldots$

$$B_q(\alpha,\beta;x) = C\,x^\alpha\frac{\theta(q^{\alpha+\beta}x)}{\theta(q^\beta x)}$$

where $\theta(x) := (x;q)_\infty(q/x;q)_\infty(q;q)_\infty$ is Jacobi's theta function and

$$C=\frac{\Gamma_q(\alpha)\Gamma_q(1-\alpha-\beta)}{\Gamma_q(1-\beta)}.$$

This is equivalent to **Ramanujan's ${}_1\psi_1$-summation formula**

$$\sum_{\nu=-\infty}^{\infty}\frac{(a;q)_\nu}{(b;q)_\nu}x^\nu = \frac{(ax;q)_\infty(q;q)_\infty(b/a;q)_\infty(q/ax;q)_\infty}{(x;q)_\infty(b;q)_\infty(q/a;q)_\infty(b/ax;q)_\infty},\qquad \left|\frac{b}{a}\right|<|x|<1.$$

A multi-variable generalization of the Beta integral.

Dixon–Anderson integral

Dixon(1907)
Anderson(1991)

$$\int_{z_n=x_{n-1}}^{x_n} \cdots \int_{z_2=x_1}^{x_2} \int_{z_1=x_0}^{x_1} \prod_{i=1}^{n} \prod_{j=0}^{n} |z_i - x_j|^{s_j-1} \prod_{1\le k<l\le n} (z_l - z_k)\, dz_1 dz_2 \cdots dz_n$$

$$= \frac{\Gamma(s_0)\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_0+s_1+\cdots+s_n)} \prod_{0\le i<j\le n} (x_j - x_i)^{s_i+s_j-1}$$

In 1991 Anderson gave a new proof of the Selberg integral using the above integral.

c.f. **Selberg integral** Selberg(1942)

$$\int_{[0,1]^n} \prod_{i=1}^{n} z_i^{\alpha-1}(1-z_i)^{\beta-1} \prod_{1\le j<k\le n} |z_k - z_j|^{2\gamma}\, dz_1 dz_2 \cdots dz_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\gamma j+1)\Gamma(\alpha+(n-j)\gamma)\Gamma(\beta+(n-j)\gamma)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+(n+j-2)\gamma)}$$

q-Dixon–Anderson integral

Evans(1992)

$$\int_{z_n=x_{n-1}}^{x_n}\cdots\int_{z_2=x_1}^{x_2}\int_{z_1=x_0}^{x_1}\prod_{i=1}^{n}\prod_{j=0}^{n}(qz_i/x_j;q)_{s_j-1}\prod_{1\le k<l\le n}(z_l-z_k)\,d_qz_1d_qz_2\cdots d_qz_n$$

$$=\frac{\Gamma_q(s_0)\Gamma_q(s_1)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0+s_1+\cdots+s_n)}\prod_{0\le i<j\le n}x_j(x_i/x_j;q)_{s_j}(qx_j/x_i;q)_{s_i-1}$$

As another known result, **Milne–Gustafson's sum**

Milne(1985)
Gustafson(1987)

$$\sum_{(\nu_1,\dots,\nu_n)\in\mathbb{Z}^n}t^{\nu_1+\cdots+\nu_n}\prod_{i,j=1}^{n}\frac{(a_ix_j/x_i;q)_{\nu_j}}{(b_ix_j/x_i;q)_{\nu_j}}\prod_{1\le i<j\le n}\frac{x_iq^{\nu_i}-x_jq^{\nu_j}}{x_i-x_j}.$$

$$=\frac{(a_1\cdots a_nt;q)_\infty(q/a_1\cdots a_nt;q)_\infty}{(t;q)_\infty(q^{1-n}b_1\cdots b_n/a_1\cdots a_nt;q)_\infty}$$
$$\times\prod_{i,j=1}^{n}\frac{(b_ix_j/a_jx_i;q)_\infty(qx_i/x_j;q)_\infty}{(qx_i/a_ix_j;q)_\infty(b_ix_j/x_i;q)_\infty}.$$

The aim is to give a summation theorem including both of the above formulae.

Multi-variable (bilateral) Jackson integral

For $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$ and $f(z) = f(z_1, \ldots, z_n)$ on $(\mathbb{C}^*)^n$, define

$$\int_0^{x\infty} f(z)\,\frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n} := (1-q)^n \sum_{(\nu_1,\ldots,\nu_n)\in\mathbb{Z}^n} f(x_1q^{\nu_1}, \ldots, x_nq^{\nu_n}).$$

Set $a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1} \in \mathbb{C}^*$ satisfying $q < |a_1a_2\cdots a_{n+1}b_1b_2\cdots b_{n+1}|$.

Jackson integral of Dixon-Anderson type

$$J(x) := \int_0^{x\infty} \Phi(z)\Delta(z)\,\frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n}$$

converges absolutely, where

$$\Phi(z) := z_1z_2\cdots z_n\prod_{i=1}^{n}\prod_{j=1}^{n+1}\frac{(qa_j^{-1}z_i;q)_\infty}{(b_jz_i;q)_\infty}, \quad \Delta(z) := \prod_{1\le i<j\le n}(z_j-z_i).$$

Remark. If we substitute a_j and b_j as $a_j \to x_{j-1}$ and $b_j \to q^{s_{j-1}}/x_{j-1}$,

$$\frac{\Phi(z)\Delta(z)}{z_1z_2\cdots z_n} \longrightarrow \prod_{i=1}^{n}\prod_{j=0}^{n}\frac{(qz_i/x_j;q)_\infty}{(q^{s_j}z_i/x_j;q)_\infty}\prod_{1\le i<j\le n}(z_j-z_i),$$

which exactly coincides with the integrand of Evans's q-Dixon–Anderson integral.

Theorem (of this talk) For an arbitrary point $(x_1, x_2, \ldots, x_{n+1}) \in (\mathbb{C}^*)^{n+1}$, set

$$(\widehat{x}_i) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \in (\mathbb{C}^*)^n \quad \text{for} \quad i = 1, 2, \ldots, n+1.$$

The Jackson integral $J(x)$ as a sum over $\mathbb{Z}^n$ satisfies

$$\sum_{i=1}^{n+1} (-1)^{i-1} \underbrace{J(\widehat{x}_i)}_{\text{bilateral}} = C \frac{\theta(x_1 x_2 \cdots x_{n+1} b_1 b_2 \cdots b_{n+1})}{\prod_{i=1}^{n+1} \prod_{j=1}^{n+1} \theta(x_i b_j)} \prod_{1 \le i < j \le n+1} x_j \theta(x_i / x_j),$$

where C is a constant independent of $(x_1, x_2, \ldots, x_{n+1}) \in (\mathbb{C}^*)^{n+1}$, which is explicitly written as

$$C = (1-q)^n \frac{(q)_\infty^n \prod_{i=1}^{n+1} \prod_{j=1}^{n+1} (q a_i^{-1} b_j^{-1})_\infty}{(q a_1^{-1} \cdots a_{n+1}^{-1} b_1^{-1} \cdots b_{n+1}^{-1})_\infty}.$$

Corollary (Evans 1992) If $x = a = (a_1, a_2, \ldots, a_{n+1}) \in (\mathbb{C}^*)^{n+1}$, the Jackson integral $J(\widehat{a}_i)$ as a sum over $\mathbb{N}^n$ satisfies

$$\sum_{i=1}^{n+1} (-1)^{i-1} \underbrace{J(\widehat{a}_i)}_{\text{unilateral}} = (1-q)^n \frac{(q)_\infty^n (a_1 \cdots a_{n+1} b_1 \cdots b_{n+1})_\infty}{\prod_{i=1}^{n+1} \prod_{j=1}^{n+1} (a_i b_j)_\infty} \prod_{1 \le i < j \le n+1} a_j \theta(a_i / a_j),$$

which is equivalent to Evans's q-Dixon–Anderson integral.

Remark. The Milen–Gustafson sum is also obtained from a specialization of the above theorem.

Comments on our proof of the theorem

Evans gave a proof for his formula of the q-Dixon–Anderson integral under the restriction on all exponents s_i as positive integers. As he pointed out in his paper, by analytic continuation s_i can be considered as complex numbers after proving in the setting s_i are integers. But we wanted to start without restrictions on the parameters being integers, and then our method is based on regarding $J(x)$ as a solution of q-difference equations fixed by its asymptotic behavior.

(1) q-difference equations

For the series expansion of classical generalized hypergeometric series, a general theory is known as Mellin's method [1907] to obtain the functional equations (i.e., differential or difference equations). In the early 1990's Aomoto developed Mellin's method to the case of bilateral Jackson integrals, and actually he applied this general theory to his simple proof for the product expression of the (q-)Selberg integral. Our method to deduce the q-difference equations for the Jackson integral of Dixon–Anderson type is consistent with Aomoto's work. The explicit q-difference equations are

$$T_{a_j}J(x)=\frac{\prod_{i=1}^{n+1}(1-b_i^{-1}a_j^{-1})}{1-\prod_{i=1}^{n+1}b_i^{-1}a_i^{-1}}J(x)\quad\text{and}\quad T_{b_j}J(x)=\frac{\prod_{i=1}^{n+1}(1-a_ib_j)}{1-\prod_{i=1}^{n+1}a_ib_i}J(x)$$

for $j=1,2,\ldots,n+1$. Here T_{a_j} means the q-shift operator of $a_j\to qa_j$, i.e.,

$$T_{a_j}f(\ldots,a_j,\ldots)=f(\ldots,qa_j,\ldots).$$

(2) Asymptotic behavior

After repeated use of the q-difference equations (in this case, two-term recurrence relations), as a boundary condition for these equations, it is necessary to obtain an asymptotic behavior of the Jackson integral as parameters go to infinity. We used a discrete version of the saddle point method for the integral representation of functions. As we often experience, the critical point giving the principal term of the asymptotic behavior can sometimes be found in the imaginary cycle of the integral. In our case, to fix $x = (b_1^{-1}, \ldots, b_n^{-1})$ is corresponding to the choice of the "imaginary cycle", and then the negative power part of the extended Jackson integral contains only one term which gives the principal term.

In this sense, from our analytic point of view, to extend the Jackson integral from unilateral to bilateral is essential for both (1) the derivation of the q-difference equations and (2) the asymptotic behavior of $J(x)$.