On highest weight representations of conformal Galilei algebras

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Outline

- Some motivation and definition of conformal Galilei algebras
- Singular vectors and irreducible modules
- Kac determinants
- Summary and future work

Infinitesimal transformations in $d+1$ dimensional space-time

(Negro, del Olmo, Rodriguez-Marco 1997; Martelli, Tachikawa 2010)

$$\ell \in \frac{1}{2}\mathbb{Z}, \quad i,j = 1,2,\ldots,d$$

- Rotation $J_{ij} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}$; $(x_i \longrightarrow x_i + \epsilon x_j,\ x_j \longrightarrow x_j - \epsilon x_i)$
- Time translation $H = \frac{\partial}{\partial t}$ $(t \longrightarrow t + \epsilon)$
- Dilatation $D = -2t\frac{\partial}{\partial t} - 2\ell x_i \frac{\partial}{\partial x_i}$ $(t \longrightarrow (1-2\epsilon)t,\ x_i \longrightarrow (1-2\ell\epsilon)x_i)$
- Conformal transformation
  $C = t^2\frac{\partial}{\partial t} + 2\ell t x_i \frac{\partial}{\partial x_i}$ $(t \longrightarrow (1+\epsilon t)t,\ x_i \longrightarrow (1+2\ell\epsilon t)x_i)$
- $P_{n,i} = (-t)^n \frac{\partial}{\partial x_i}$, $n = 0,1,\ldots,2\ell$
  - $n=0$ (translation) $(x_i \longrightarrow x_i + \epsilon)$
  - $n=1$ (Galilean boost) $(x_i \longrightarrow x_i - \epsilon t)$
  - $n=2$ (acceleration) … $(x_i \longrightarrow x_i + \epsilon t^2)$

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Non-trivial commutation relations

$$[D,H]=2H,\quad [D,C]=-2C,\quad [C,H]=D,\quad (sl(2))$$

$$[J_{ij},J_{k\ell}]=\delta_{ik}J_{j\ell}+\delta_{j\ell}J_{ik}-\delta_{i\ell}J_{jk}-\delta_{jk}J_{i\ell},\quad (so(d))$$

$$[H,P_{n,i}]=-nP_{n-1,i},\quad [D,P_{n,i}]=2(\ell-n)P_{n,i},\quad [C,P_{n,i}]=(2\ell-n)P_{n+1,i}$$

$$[J_{ij},P_{n,k}]=\delta_{ik}P_{n,j}-\delta_{jk}P_{n,i}\quad (\{P_{n,i}\}\text{ Abelian ideal})\ \Rightarrow\ \text{non-semisimple.}$$

Abelian ⇒ solvable ⇒ nontrivial maximal solvable ideal
(semisimple ⇔ trivial maximal solvable ideal)

Central extensions

There are two types of central extensions:

(1) For any d, $\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$

$$[P_{m,i}, P_{n,j}] = I_{mn}\delta_{ij}M$$

where M is central, I_{mn} is antisymmetric.

(2) For $d = 2$, $\ell = 1, 2, 3, \dots$

$$[P_{m,i}, P_{n,j}] = \hat{I}_{mn}\epsilon_{ij}\Theta$$

where Θ is central, $\hat{I}_{mn}$ is symmetric, ϵ_{ij} is antisymmetric.

(Bargmann 1954; Levy-Leblond 1972; Jackiw, Nair 2000;
Lukierski, Stichel, Zakrzewski 2006, 2007; Martelli, Tachikawa 2010)

Example: Schrödinger algebra ($\ell = \frac{1}{2}$) with central extension

(Niederer 1972; Dobrev, Doebner, Mrugalla 1997)

Differential operator realisation:

$$P_{0,j} = \frac{\partial}{\partial x_j}, \quad P_{1,j} = -t\frac{\partial}{\partial x_j} - m x_j, \quad M = m,$$
$$H = \frac{\partial}{\partial t}, \quad D = -2t\frac{\partial}{\partial t} - x_k\frac{\partial}{\partial x_k} - \frac{1}{2},$$
$$C = t^2\frac{\partial}{\partial t} + t x_j\frac{\partial}{\partial x_j} + \frac{1}{2} m x_k x_k - \frac{t}{2},$$
$$J_{ij} = -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i}.$$

$\Rightarrow$ Lie symmetries of (i) free Schrödinger equation (m pure imaginary); (ii) heat equation (m real)

$$\nabla^2\psi - 2m\frac{\partial\psi}{\partial t} = 0.$$

Invariant equations

Semisimple Lie algebras

↓

(Kostant 1975; Dobrev 1988)

↓

Canonical construction of invariant equations via a differential operator realisation.

↓

(Dobrev, Doebner, Mrugalla 1997)

↓

Generalised to the (non-semisimple) centrally extended Schrödinger algebra.

This method requires knowledge of *singular vectors*.

Non-semisimple Lie algebras

Representation theory is well understood for semisimple Lie algebras (e.g. Weyl's theorem, theory of weights, etc.).

Representation theory of non-semisimple Lie algebras remains largely undeveloped.

- $\rightarrow$ In some cases it is possible to study highest weight submodules of the Verma modules using a triangular decomposition of the Lie algebra consistent with the triangular decomposition of its semisimple part.
- $\rightarrow$ Conjecture (Dobrev, Doebner, Mrugalla 1997): A theory of highest weight modules can be developed for arbitrary non-semisimple Lie algebras with such a triangular decomposition.
- $\rightarrow$ Requires knowledge of *singular vectors*.

$\Rightarrow$ Motivation is twofold: (1) invariant equations, (2) highest weight modules

$\mathfrak{g}_\ell$: Conformal Galilei algebra with central extension, $d = 1$, $\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$ (Aizawa, PSI, Kimura 2012):

Basis $\{C, D, H, P_n \mid n = 0, 1, 2, \cdots, 2\ell\}$

$$\begin{aligned}
&[D, H] = 2H, && [D, C] = -2C, && [C, H] = D, \\
&[H, P_n] = -nP_{n-1}, && [D, P_n] = 2(\ell - n)P_n, && [C, P_n] = (2\ell - n)P_{n+1}, \\
&[P_m, P_n] = I_{m,n}M, && I_{m,n} = \delta_{m+n,2\ell}(-1)^{m+\ell+\frac{1}{2}} m!(2\ell - m)!
\end{aligned}$$

Triangular decomposition:

$$\begin{aligned}
\mathfrak{g}_\ell^+ &= \left\{ H,\ P_0,\ P_1,\ \cdots,\ P_{\ell-\frac{1}{2}} \right\} \\
\mathfrak{g}_\ell^0 &= \{ D,\ M \} \\
\mathfrak{g}_\ell^- &= \left\{ C,\ P_{\ell+\frac{1}{2}},\ P_{\ell+\frac{3}{2}},\ \cdots,\ P_{2\ell} \right\}.
\end{aligned}$$

Verma module

Let $|\delta,\mu\rangle$ be a highest weight vector of the Verma module $V^{\delta,\mu}$, such that

$$D\,|\delta,\mu\rangle = \delta\,|\delta,\mu\rangle\,, \qquad M\,|\delta,\mu\rangle = \mu\,|\delta,\mu\rangle\,, \qquad X\,|\delta,\mu\rangle = 0, \quad X \in \mathfrak{g}_\ell^+,$$

with $V^{\delta,\mu}$ being determined by $U(\mathfrak{g}_\ell^-)\,|\delta,\mu\rangle$.

$\Rightarrow$ basis of $V^{\delta,\mu}$ is

$$\left\{ C^h \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j}\,|\delta,\mu\rangle \;\middle|\; h, k_0, k_1, \ldots, k_{\ell-\frac{1}{2}} \in \mathbb{Z}_+ \right\}.$$

Eigenvalue of D corresponding to $C^h \prod_{j=0}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j}\,|\delta,\mu\rangle$ is $\delta - 2h - \sum_{j=0}^{\ell-\frac{1}{2}}(2j+1)k_j$.

$\Rightarrow$ Define "level" m within $V^{\delta,\mu}$ by $m = 2h + \sum_{j=0}^{\ell-\frac{1}{2}}(2j+1)k_j$.

Notation

For a fixed level m, since $k_0 = m - 2h - \sum_{j=1}^{\ell-\frac{1}{2}} (2j+1)k_j$, we find it convenient to denote the basis vectors at level m by

$$|h, \underset{\sim}{k}; m\rangle = C^h P_{\ell+\frac{1}{2}}^{m-2h-\sum_{j=1}^{\ell-\frac{1}{2}}(2j+1)k_j} \prod_{j=1}^{\ell-\frac{1}{2}} P_{\ell+\frac{1}{2}+j}^{k_j} |\delta, \mu\rangle \,,$$

with $\underset{\sim}{k} = (k_1, k_2, \cdots, k_{\ell-\frac{1}{2}})$.

We have

$$V^{\delta,\mu} = \bigoplus_{m\in\mathbb{Z}_+} V_m^{\delta,\mu},$$

where $V_m^{\delta,\mu}$ is the space spanned by the vectors $|h, \underset{\sim}{k}; m\rangle$ for fixed m.

Singular vectors

For $m > 0$, a *singular vector* $|u_m\rangle$ is defined as an element of $V_m^{\delta,\mu}$ satisfying

$$\mathfrak{g}_\ell^+ \, |u_m\rangle = 0, \quad \mathfrak{g}_\ell^+ = \left\{ \, H, \; P_0, \; P_1, \; \cdots, \; P_{\ell-\frac{1}{2}} \, \right\}.$$

Theorem

(1) If $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$ for $q \in \mathbb{Z}^+$ then the following is a singular vector at level $2q$:

$$|u_{2q}\rangle = \left(\alpha_\ell \mu \, C - P^2_{\ell+\frac{1}{2}}\right)^q |\delta, \mu\rangle \in V_{2q}^{\delta,\mu}$$

where $\alpha_\ell = 2((\ell - \frac{1}{2})!)^2$.

(2) In order for the vector $|u_m\rangle = \sum_{h,\underset{\sim}{k}} a(h, \underset{\sim}{k}) \, |h, \underset{\sim}{k}; m\rangle$ to be a singular vector, m must be even, in which case the coefficients $a(h, \underset{\sim}{k})$ are unique up to an overall factor.

Irreducible highest weight modules

$\Rightarrow V^{\delta,\mu}$ has precisely one singular vector that exists at level $2q$ with

$$\delta = q - 1 - \frac{1}{2}\left(\ell + \frac{1}{2}\right)^2 .$$

$\Rightarrow V^{\delta,\mu}$ contains the submodule $I^{\delta,\mu} = U(\mathfrak{g}_\ell^-)\,|u_{2q}\rangle$.
Using similar arguments to part (2) of the previous theorem, one may show that there are no singular vectors in the quotient module $V^{\delta,\mu}/I^{\delta,\mu}$.

Theorem

The irreducible highest weight modules of $\mathfrak{g}_\ell$ for half odd integer ℓ with $\mu \neq 0$ are listed as follows:

- $V^{\delta,\mu}$ if $\delta \neq q - 1 - \frac{1}{2}\left(\ell + \frac{1}{2}\right)^2$,
- $V^{\delta,\mu}/I^{\delta,\mu}$ if $\delta = q - 1 - \frac{1}{2}\left(\ell + \frac{1}{2}\right)^2$,

where $q \in \mathbb{Z}^+$. All modules are infinite dimensional.

Shapovalov form

The Shapovalov form (Shapovalov 1972) (,) on $V^{\delta,\mu}$ is defined by setting

$$(|\delta,\mu\rangle, |\delta,\mu\rangle) \equiv \langle\delta,\mu|\delta,\mu\rangle = 1,$$

and

$$(A|x\rangle, B|y\rangle) = (|x\rangle, \omega(A)B|y\rangle), \quad \forall\, |x\rangle, |y\rangle \in V^{\delta,\mu},\ A, B \in \mathfrak{g}_\ell,$$

where ω is an algebra anti-automorphism defined by

$$\omega(P_j) = P_{2\ell-j},\ \omega(C) = H,\ \omega(H) = C,\ \omega(D) = D,\ \omega(M) = M.$$

This form is symmetric when restricted to the basis $\{|h, \underset{\sim}{k}; m\rangle\}$ of $V_m^{\delta,\mu}$.

Matrix of Shapovalov forms at level m

Given an ordered basis $\{v_i\}$ of $V_m^{\delta,\mu}$, define a matrix whose entry in the ith row and jth column is the number (v_i, v_j).

e.g. $V_2^{\delta,\mu}$ has basis $\left\{ v_1 = |0, \underset{\sim}{0}; 2\rangle = P^2_{\ell+\frac{1}{2}} |\delta, \mu\rangle\,, \quad v_2 = |1, \underset{\sim}{0}; 2\rangle = C\,|\delta, \mu\rangle \right\}$

$$\Rightarrow \quad \begin{pmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_2, v_1) & (v_2, v_2) \end{pmatrix} = \begin{pmatrix} 2((\ell - \frac{1}{2})!(\ell + \frac{1}{2})!)^2\mu^2 & ((\ell + \frac{1}{2})!)^2\mu^2 \\ ((\ell + \frac{1}{2})!)^2\mu^2 & -\delta \end{pmatrix}$$

The null space of such a matrix gives a set of *null vectors*, that are orthogonal to the basis vectors.

- Null vectors exist iff determinant is zero.
- Singular vectors are null vectors.
- Descendents of null vectors are also null vectors.

Kac determinant

The determinant of the matrix of Shapovalov forms is referred to as the *Kac determinant* at level m, denoted $\mathcal{D}_m^\ell$.

e.g. For $m = 2$, $\mathcal{D}_m^\ell = -\mu^2((\ell - \frac{1}{2})!(\ell + \frac{1}{2})!)^2(2\delta + (\ell + \frac{1}{2})^2)$

Recall the **Theorem** on existence of singular vectors:

If $2\delta - 2(q-1) + (\ell + \frac{1}{2})^2 = 0$ for $q \in \mathbb{Z}^+$ then there exists a singular vector at level $2q$.

Conjecture (Dobrev, Doebner, Mrugalla 1997): For $\ell = \frac{1}{2}$,

$$\mathcal{D}_m^{1/2} = C_m \mu^{\alpha_m} \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} (2\delta - 2j + 1)^{\lfloor \frac{m}{2} \rfloor - j}$$

where

$$\alpha_m = \begin{cases} \frac{1}{4} m(m+2); & m \text{ even} \\ \frac{1}{4}(m+1)^2; & m \text{ odd.} \end{cases}$$

Dimension of $V_m^{\delta,\mu}$

Vectors in the basis $\{|h, \underset{\sim}{k}; m\rangle\}$ are in one to one correspondence with the restricted set of integer partitions of the integer m, with parts taken from the subset of integers

$$\{2\} \cup \left\{2j+1 \mid j = 0, 1, \dots, \ell - \frac{1}{2}\right\}.$$

Vectors in the basis of $V_m^{\delta,\mu}$ can be enumerated by

$$\left\{(h, k_1, \dots, k_j, \dots, k_{\ell-\frac{1}{2}}) \;\middle|\; 0 \leq h \leq \left\lfloor \frac{m}{2} \right\rfloor, \; 0 \leq k_{\ell-\frac{1}{2}} \leq \left\lfloor \frac{m-2h}{2(\ell-\frac{1}{2})+1} \right\rfloor, \dots \right.$$
$$\dots 0 \leq k_j \leq \left\lfloor \frac{m - 2h - \sum_{n=j+1}^{\ell-\frac{1}{2}}(2n+1)k_n}{2j+1} \right\rfloor, \; \dots,$$
$$\left. \dots 0 \leq k_1 \leq \left\lfloor \frac{m - 2h - \sum_{n=2}^{\ell-\frac{1}{2}}(2n+1)k_n}{3} \right\rfloor \right\}.$$

Dimension of $V_m^{\delta,\mu}$

$\Rightarrow \dim(V_m^{\delta,\mu}) \equiv d_m^\ell$ given by

$$\Rightarrow \quad d_m^\ell = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_{\ell-\frac{1}{2}}=0}^{\left\lfloor \frac{m-2h}{2(\ell-\frac{1}{2})+1} \right\rfloor} \cdots \sum_{k_j=0}^{\left\lfloor \frac{m-2h-\sum_{n=j+1}^{\ell-\frac{1}{2}}(2n+1)k_n}{2j+1} \right\rfloor} \cdots \sum_{k_1=0}^{\left\lfloor \frac{m-2h-\sum_{n=2}^{\ell-\frac{1}{2}}(2n+1)k_n}{3} \right\rfloor} 1.$$

e.g.

$$d_m^{1/2} = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} 1, \qquad d_m^{5/2} = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{m-2h}{5} \rfloor} \sum_{k_1=0}^{\lfloor \frac{m-2h-5k_2}{3} \rfloor} 1,$$

$$d_m^{3/2} = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_1=0}^{\lfloor \frac{m-2h}{3} \rfloor} 1, \qquad d_m^{7/2} = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_3=0}^{\lfloor \frac{m-2h}{7} \rfloor} \sum_{k_2=0}^{\lfloor \frac{m-2h-7k_3}{5} \rfloor} \sum_{k_1=0}^{\lfloor \frac{m-2h-7k_3-5k_2}{3} \rfloor} 1.$$

Dimension of $V_m^{\delta,\mu}$

Characterise d_m^ℓ by the generating function

$$F^\ell(x) = \frac{1}{1-x^2}\prod_{j=0}^{\ell-\frac{1}{2}}\frac{1}{1-x^{2j+1}},$$

in the sense that the coefficients of the formal power series are the numbers d_m^ℓ, i.e.

$$F^\ell(x) = \sum_{m=0}^{\infty} d_m^\ell x^m.$$

e.g.

$$d_m^{1/2} = \left\lfloor \frac{m+2}{2} \right\rfloor, \qquad d_m^{5/2} = \left\lfloor \frac{2m^3+33m^2+162m+360}{360} \right\rfloor,$$

$$d_m^{3/2} = \left\lfloor \frac{m^2+6m+12}{12} \right\rfloor, \qquad d_m^{7/2} = \left\lfloor \frac{m^4+36m^3+442m^2+2124m+5040}{5040} \right\rfloor.$$

Dependence on δ

- Singular vectors are null vectors & we count descendants

$\Rightarrow$ $\mathcal{D}^{\ell}_{m}$ contains the factor $\left(2\delta - 2(q-1) + \left(\ell + \frac{1}{2}\right)^2\right)^{d^{\ell}_{m-2q}}$, for every integer $q > 0$ satisfying $m \geq 2q$.

- For $h \leq h'$, $\langle h, \underset{\sim}{k}; m | h', \underset{\sim}{k}'; m\rangle \sim \delta^h$

$\Rightarrow$ Diagonal entries contain polynomials in δ of maximal degree

$\Rightarrow$ $\mathcal{D}^{\ell}_{m}$ is a polynomial in δ of degree given by the number of C generators in all basis vectors.

- The number of vectors in the basis of $V^{\delta,\mu}_{m}$ containing C^h must be $O^{2\ell}_{m-2h}$, where $O^{2\ell}_{n}$ is the number of integer partitions of n comprising only odd parts no greater than 2ℓ.

$\Rightarrow$ Number of C generators in all basis vectors is $\displaystyle\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} n O^{2\ell}_{m-2n}$.

Dependence on δ

One may show

$$\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} n O_{m-2n}^{2\ell} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} d_{m-2(j+1)}^{\ell},$$

RHS = sum of powers of the previously obtained factors.
$\Rightarrow \mathcal{D}_m^{\ell}$ is of the form

$$\mathcal{D}_m^{\ell} = f(\mu) \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \left(2\delta - 2j + \left(\ell + \frac{1}{2} \right)^2 \right)^{d_{m-2(j+1)}^{\ell}}$$

Dependence on μ

For a vector $v \equiv |h, \underset{\sim}{k}; m\rangle$ in the basis $\gamma = \{|h, \underset{\sim}{k}; m\rangle\}$ of $V_m^{\delta,\mu}$, define the *μ-weight of* v, denoted ρ_v, as

$$\rho_v = m - 2\left(h + \sum_{j=1}^{\ell-\frac{1}{2}} jk_j\right).$$

Note that ρ_v is the sum of powers of all the P_n-type generators appearing in v.

- For all $v, w \in \gamma$, either $(v, w) = 0$ or $(v, w) = Y\mu^{\frac{1}{2}(\rho_v+\rho_w)}$.

$\Rightarrow$ $\mathcal{D}_m^\ell = Z\mu^{\sum_{v\in\gamma}\rho_v}$.

- Let $e_m^\ell = \sum_{v\in\gamma}\rho_v$, which gives the total number of P_n-type generators that occur in the basis γ.

$\Rightarrow$

$$e_m^\ell = \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k_{\ell-\frac{1}{2}}=0}^{\left\lfloor \frac{m-2h}{2(\ell-\frac{1}{2})+1} \right\rfloor} \cdots \sum_{k_j=0}^{\left\lfloor \frac{m-2h-\sum_{n=j+1}^{\ell-\frac{1}{2}}(2n+1)k_n}{2j+1} \right\rfloor} \cdots \sum_{k_1=0}^{\left\lfloor \frac{m-2h-\sum_{n=2}^{\ell-\frac{1}{2}}(2n+1)k_n}{3} \right\rfloor} \left(m - 2\left(h + \sum_{j=1}^{\ell-\frac{1}{2}} jk_j\right)\right).$$

Dependence on μ, Kac determinant

Generating function given by

$$E^\ell(x) = \sum_{m=0}^{\infty} e_m^\ell x^m = \left(\sum_{i=0}^{\ell-\frac{1}{2}} \frac{x^{2i+1}}{1-x^{2i+1}} \right) \frac{1}{1-x^2} \prod_{j=0}^{\ell-\frac{1}{2}} \frac{1}{1-x^{2j+1}}.$$

Theorem

$$\mathcal{D}_m^\ell = C_m^\ell \mu^{e_m^\ell} \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor -1} \left(2\delta - 2j + \left(\ell + \frac{1}{2} \right)^2 \right)^{d_{m-2(j+1)}^\ell},$$

for some constant C_m^ℓ.

e.g. $\ell = 1/2$

$$
\begin{aligned}
e_m^{1/2} &= \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} (m - 2h) = \left(m - \left\lfloor \frac{m}{2} \right\rfloor\right) \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \\
&= \begin{cases} \frac{1}{4} m(m+2); & m \text{ even} \\ \frac{1}{4}(m+1)^2; & m \text{ odd.} \end{cases}
\end{aligned}
$$

and

$$
d_{m-2(j+1)}^{1/2} = \left\lfloor \frac{m - 2(j+1) + 2}{2} \right\rfloor = \left\lfloor \frac{m - 2j}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor - j.
$$

$\Leftrightarrow$ conjecture of Dobrev, Doebner, Mrugalla.

Future work

- Invariant equations for $\mathfrak{g}_\ell$? $\longrightarrow$ forthcoming work of Aizawa, Segar, Kimura
- Infinite dimensional extensions?

  $\longrightarrow$ $\ell = 1/2$ work on Schrödinger-Virasoro algebras by Roger, Unterberger among others
  $\longrightarrow$ $\ell > 1/2$? Kimura's thesis?
- Quantum group analogues?

  $\longrightarrow$ $\ell = 1/2$ Dobrev, Doebner, Mrugalla
  $\longrightarrow$ $\ell > 1/2$?
- Other non-semisimple Lie algebras related to Schrödinger equation with potentials? e.g. "Newton-Hooke algebras" relate to the simple harmonic oscillator.
- Development of more general representation theory related to $\mathbb{Z}$-graded algebras?