"[Moduli spaces] have also appeared in theoretical physics like string theory: many computations of path integrals are reduced to integrals of Chern classes on such moduli spaces."

Shing-Tung Yau

Phants and Surfaces

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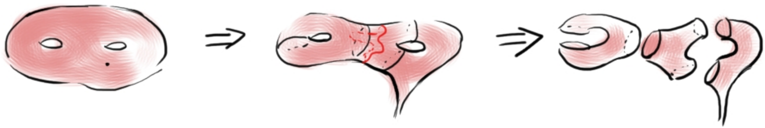
How can we study it?

- build every Riemann surface.
- identify biholomorphic ones.

The anatomy of a Riemann surface:

- A Riemann surface $R \Rightarrow$ a unique hyperbolic surface R_h.



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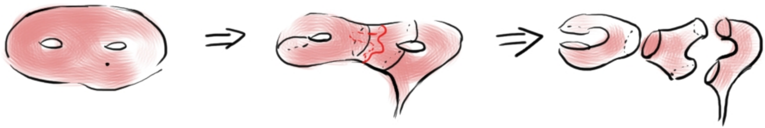

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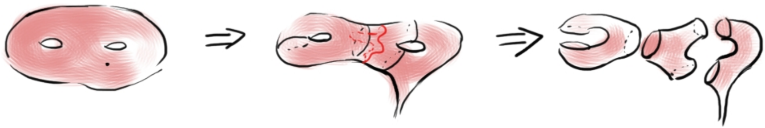
- A Riemann surface $R \Rightarrow$ a unique hyperbolic surface R_h.
- Simple loops on hyperbolic surfaces $\Rightarrow$ unique geodesics.
- Cut along these geodesics $\Rightarrow$ hyperbolic pairs of pants.



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- vary the gluing.

Teichmüller space

Take geodesics $\{\gamma_1, \ldots, \gamma_{3g-3+n}\}$ that decompose R_h into pairs of pants. For each γ_i, we have:

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We get the Teichmüller space

$$\mathcal{T}(S) = (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}.$$

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$$\mathcal{T}(S) \to \mathcal{M}(S)$$

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The Weil-Petersson 2-form

$$\Omega_{WP} := \mathrm{d}\ell_1 \wedge \mathrm{d}\tau_1 + \ldots + \mathrm{d}\ell_{3g-3+n} \wedge \mathrm{d}\tau_{3g-3+n}$$

is invariant under this group action, and makes $\mathcal{M}(S)$ a symplectic manifold.

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- Shove the coefficients in a generating function and exponentiate to get a solution to the KdV equations!

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Up to cone-angle π, everything we've talked about still holds true.

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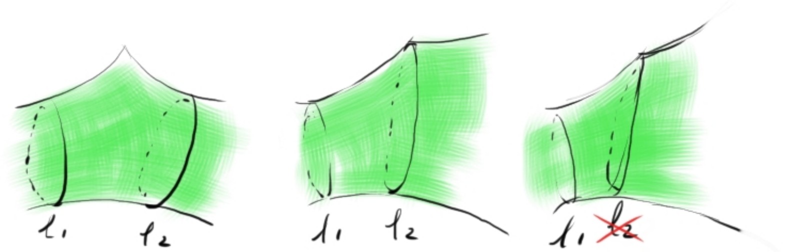
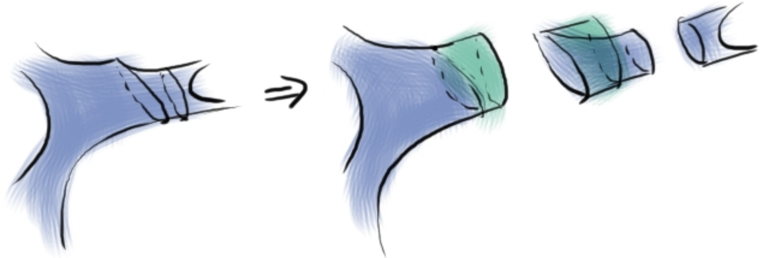


To fix this: *push past* the cone-point.

When cutting surface along given broken geodesics, extend (and sometimes retract) our surface to obtain *phantom pants* or *phants* with geodesic boundary.



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Our phants coordinates are very representation theoretic, so these coordinates willl produce the "correct" Weil-Petersson form.