Integrable dynamics: practical applications and abstract theory

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Different types of dynamics (rough division)

- Linear systems: Modes move independently (dispersion)
- Nonlinear and integrable Few dominant modes, interacting constructively (coherence, e.g., solitons)
- Nonlinear and chaotic Few dominant modes, expanding differences (e.g., the butterfly effect).
- Turbulence: Lost of contributing modes.

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I will illustrate this with three examples:

- Hamiltonian mechanical systems and Liouville integrability
- Solitonic wave equations and multisoliton solutions.
- Lattice equations and multidimensional consistency.

Finally we will take an “overview” on integrable systems.

1. Hamiltonian dynamics/Liouville integrability

Hamiltonian dynamics

To define a Hamiltonian mechanical system we need

- N coordinates q_j, N momenta p_j ($= 2N$ dim. *phase space*)
- Hamiltonian function $H = H(p, q)$
- Poisson bracket, for example

$$\{A, B\} = \sum_{j=1}^{N} \left(\frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial q_j} \frac{\partial A}{\partial p_j} \right)$$

Then the equations of motion are given by

$$\tfrac{d}{dt} q_j = \{q_j, H\}, \quad \tfrac{d}{dt} p_j = \{p_j, H\}, \quad j = 1, \ldots, N$$

Liouville integrability

Definition: A Hamiltonian dynamical system defined by $H(p,q)$ (in $2N$ dimensional phase space) is Liouville integrable, if there are N functions $I_k(p,q)$ (H one of them) such that the I_k

1. are functionally independent,
2. are in involution, i.e., $\{I_n, I_m\} = 0$, $\forall n, m$,
3. are globally defined regular functions.

Since the I_k are conserved, the motion stays on the intersection of the level sets $I_k(p,q) =$const. $k = 1, ..., N$.

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Furthermore, **IF** the motion is compact **THEN** the phase space is foliated into tori.

Theorem of Kolmogorov-Arnold-Moser:
Most of the tori persist under small perturbations.

Scattering systems (N-particles on a line)

Calogero-Moser system (F. Calogero 1971, J. Moser 1975):

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{j=1}^{N-1} \frac{1}{(q_{j+1}-q_j)^2}$$

Generalizations include replacing the $1/q^2$ potential by suitable trigonometric or elliptic functions.

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Toda lattice (molecule) (M. Toda 1967)

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} e^{(q_{j+1}-q_j)}$$

Both models are integrable for any N.

There are also generalizations to various Lie Algebras (Olshanetsky and Perelomov, Phys. Rep. **71**, 313 (1981).)

Extension and generalization:

- Motion on a manifold
- Different Poisson bracket (it must be antisymmetric, act as a derivative on products, and satisfy Jacobi identity)

There are practically no classification results, but many isolated examples are known. (for $N = 2$, see JH, Phys. Rep. (1987))

For homogeneous potentials some classification results have been obtained using Galois theory. (Ziglin, Yoshida, Morales-Ruiz, Ramis, Maciejewski, Przybylska)

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Quantum versions exists (e.g. the Calogero-Sutherland model and the quantum Toda chain/lattice).

Related question: $N + 1$ (!) commuting differential operators ($p \to -i\hbar\partial_q$). (Burchnal and Chaundy (1920's), Chalykh and Veselov (since 1990), JH (1998))

The Toda lattice / the conserved quantities

Let us return to the Toda lattice

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} e^{-(q_{j+1}-q_j)}$$

H. Flaschka (1974): change variables

$$a_n = \tfrac{1}{2} e^{-\frac{1}{2}(q_{n+1}-q_n)}, \quad b_n = -\tfrac{1}{2} p_n$$

Then the Toda equations become ($a_0 = a_N = 0$)

$$\tfrac{d}{dt} a_n = a_n(b_{n+1} - b_n), \quad \tfrac{d}{dt} b_n = 2(a_n^2 - a_{n-1}^2), n = 1, \ldots, N.$$

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We would like to write these equations as a Lax pair

$$\left\{ \begin{array}{rcl} A\psi & = & \lambda\psi, \\ B\psi & = & \partial_t \psi \end{array} \right. \quad \text{consistency} \quad \Rightarrow \quad \tfrac{d}{dt} A = [B, A]$$

Lax matrix for Toda

This works, if we choose the Lax matrices (molecule version)

$$A = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{N-1} & a_{N-1} \\ 0 & \dots & 0 & a_{N-1} & b_N \end{pmatrix} \quad B = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \dots & 0 & -a_{N-1} & 0 \end{pmatrix}$$

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Conserved quantities now follow easily due to $\frac{d}{dt}A = [B, A]$:

$$\text{Let} \quad I_k := Tr(A^k), \quad \text{then} \quad \frac{d}{dt} I_k = k Tr(A^{k-1}[B, A]) = 0.$$

For example, I_1=total momentum, I_2=Hamiltonian.

Also need to show that the I_k are in involution and are functionally independent.
(Flaschka 1974, Olshanetsky and Perelomov 1981).

Toda flow and eigenvalues

J. Moser (1974): Toda molecule is a scattering system and

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Therefore if A is tri-diagonal matrix with positive off-diagonal elements, then under the Toda flow it becomes diagonal as $t \to +\infty$ and the diagonal elements are its eigenvalues.

Toda flow is a method of computing eigenvalues! (Symes 1982)

2. Soliton equations

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Thus a soliton is much more than a localized solution.

Soliton solutions and Hirota's method

Since the equations are nonlinear, the construction of multisoliton solutions is complicated.

Consider the Korteweg-de Vries (KdV) equation

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In 1971 R. Hirota proposed a change of variables $u \to F$

$$u = 2\partial_x^2 \log F.$$

where the new dependent variable F is very regular, therefore it should be a good variable to use.

Example: KdV

Substituting

$$u = 2\partial_x^2 \log F,$$

into the KdV-equation $u_t + u_{xxx} + 6uu_x = 0$, and integrating once yields a "bilinear" equation.

$$(D_x^4 + D_x D_t) F \cdot F = 0,$$

where Hirota's derivative operator D is defined by

$$\begin{aligned} D_x^n f \cdot g &= (\partial_{x_1} - \partial_{x_2})^n f(x_1) g(x_2) \big|_{x_2 = x_1 = x} \\ &\equiv \partial_y^n f(x+y) g(x-y) \big|_{y=0}. \end{aligned}$$

Like Leibniz-rule, except for the sign.

(Bilinearization of a given nonlinear PDE is not always easy.)

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Example: The bilinear equation

$$P(\vec{D}_x)F \cdot F = 0$$

has, for *any* function P, the one-soliton solution

$$F = 1+e^{\eta_1},\ \eta_\nu = \vec{x}\cdot\vec{p}_\nu+\eta_\nu^0, \quad u = 2\partial_x^2 \log(1+e^{\eta_1}) = \frac{p_1^2/2}{\cosh^2(\eta_1/2)}$$

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It also has a two-soliton solution:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2},$$

where the phase factor is

$$A_{\nu\mu} = -\frac{P(\vec{p}_\nu - \vec{p}_\mu)}{P(\vec{p}_\nu + \vec{p}_\mu)}.$$

The 3-soliton condition

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Among the KdV type equations $P(D)F \cdot F = 0$ (with $P(D)$ a polynomial in any number of variables) only the following equations (and their reductions) pass the 3SC test (J.H. 1987)

$$
\begin{aligned}
(D_x^4 - 4D_xD_t + 3D_y^2)F \cdot F &= 0, \\
(D_x^3D_t + aD_x^2 + D_tD_y)F \cdot F &= 0, \\
(D_x^4 - D_xD_t^3 + aD_x^2 + bD_xD_t + cD_t^2)F \cdot F &= 0, \\
(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y)F \cdot F &= 0.
\end{aligned}
$$

These equations also have 4SS and pass the Painlevé test.

The bilinear identity

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In this approach the center stage is taken by the bilinear identity, which is a kind of generating functional for a hierarchy:

$$\boxed{\oint \frac{dk}{2\pi i}\, e^{\xi(x-x',k)}\, \tau(x-\varepsilon(k^{-1}))\, \tau(x'+\varepsilon(k^{-1})) = 0,}$$

The variables x, x' are infinite dimensional, $x = (x_1, x_2, \dots)$,

$$\xi(x,k) = \sum_{n=1}^{\infty} x_n\, k^n, \quad \varepsilon(a) = (a, \tfrac{1}{2}a^2, \tfrac{1}{3}a^3, \cdots).$$

τ is the tau-function, corresponding to Hirota's F.

From identity to equations

$$\oint \frac{dk}{2\pi i}\, e^{\xi(x-x',k)}\, \tau(x-\varepsilon(k^{-1}))\, \tau(x'+\varepsilon(k^{-1})) = 0,$$

Change variables: $x = z + y, x' = z - y$ and express the shifts in τ using Hirota's bilinear operator D

$$\mathrm{Res}_k \left[e^{\sum_{n=1}^{\infty} 2k^n y_n}\, e^{\sum_{n=1}^{\infty}\left(y_n - \frac{1}{n\,k^n}\right) D_{z_n}}\, \tau(z)\cdot\tau(z) \right] = 0.$$

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$$\boxed{\mathrm{Res}_k\left[e^{\sum_{n=1}^{\infty}2k^n y_n}\, e^{\sum_{n=1}^{\infty}(y_n-\frac{1}{n\,k^n})D_{z_n}}\,\tau(z)\cdot\tau(z)\right]=0.}$$

Expand this in powers of y_n, choose some monomial in y_k and from its coefficient take the k^{-1} term. If y_n has weight n then at weight 3,4,5 we get the bilinear KP equations

$$\begin{aligned}
(D_1^4-4D_1D_3+3D_2)\tau\cdot\tau &= 0,\\
(D_1^3D_2+2D_3D_2-3D_1D_4)\tau\cdot\tau &= 0,\\
(D_1^6-20D_1^3D_3-80D_3^2+144D_1D_5-45D_1^2D_2^2)\tau\cdot\tau &= 0,\\
(D_1^6+4D_1^3D_3-32D_3^2-9D_1^2D_2^2+36D_2D_4)\tau\cdot\tau &= 0.
\end{aligned}$$

N-soliton solutions

Furthermore the bilinear identity

$$\oint \frac{dk}{2\pi i}\, e^{\xi(x-x',k)}\, \tau(x-\varepsilon(k^{-1}))\, \tau(x'+\varepsilon(k^{-1})) = 0,$$

(and hence all the generated equations) have
N-soliton solutions of the form

$$\tau(x) = \sum_{J\subset I} \left(\prod_{i\in J} c_i\right) \left(\prod_{i<j\in J} A_{ij}\right) \exp\left(\sum_{i\in J} \xi_i(x)\right)$$

where $I = \{1, 2, \ldots, N\}$ and

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}, \quad \xi_i(x) = \sum_{n=1}^{\infty} (p_i^n - q_i^n) x_n = \xi(x, p_i) - \xi(x, q_i).$$

Reductions

• 1+1 dimensional equations and solutions are obtained by reductions:

KdV is obtained as a 2-reduction $p^2 = q^2$ (i.e. $q = -p$), this eliminates the x_{2n} terms

Boussinesq hierarchy is obtained as a 3-reduction $p^3 = q^3$.

• There are multicomponent generalisations

• There are other bilinear identities, for example to generate the Lax pair.

For an introduction see the book
Miwa, Jimbo and Date: Solitons (CUP, 2000)

3. Lattice systems

Why discrete?

- Many mathematical constructs can be interpreted as difference relations, e.g., recursion relations.
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About discretization: There are many ways to discretize the derivative, therefore many discretized versions of familiar continuous equations.

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but even before that: What is the definition of integrability?

"Universal" definition: low growth of complexity under iterations.

There are also practical and operational definitions depending on the class of equations.

Quadrilateral lattice equations

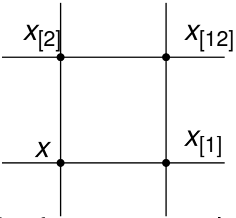
Various notations

$x_{n,m} = x_{00} = x$

$x_{n+1,m} = x_{10} = x_{[1]} = \bar{x}$

$x_{n,m+1} = x_{01} = x_{[2]} = \tilde{x}$

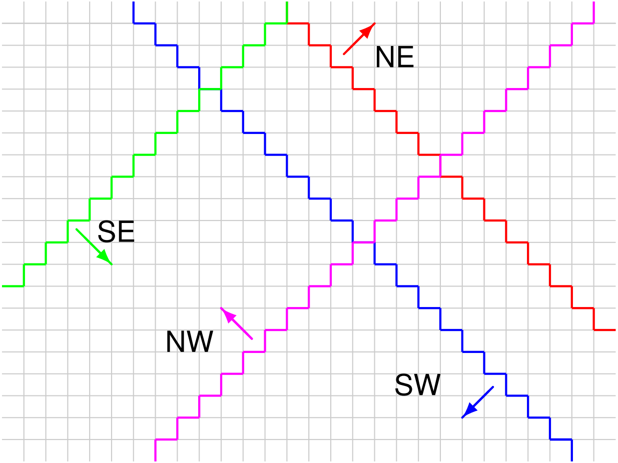
$x_{n+1,m+1} = x_{11} = x_{[12]} = \bar{\tilde{x}}$



The map is defined by relating the four corner values with a multi-linear relation:

$$k\, xx_{[1]}x_{[2]}x_{[12]} + l_1\, xx_{[1]}x_{[2]} + l_2\, xx_{[1]}x_{[12]} + l_3\, xx_{[2]}x_{[12]} + l_4\, x_{[1]}x_{[2]}x_{[12]}$$
$$+ p_1\, xx_{[1]} + p_2\, x_{[1]}x_{[2]} + p_3\, x_{[2]}x_{[12]} + p_4\, x_{[12]}x + p_5\, xx_{[2]} + p_6\, x_{[1]}x_{[12]}$$
$$+ q_1\, x + q_2\, x_{[1]} + q_3\, x_{[2]} + q_4\, x_{[12]} + u \equiv Q(x, x_{[1]}, x_{[2]}, x_{[12]}) = 0.$$

From multilinearity it follows that we can define evolution from staircase-like initial values to any direction.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CAC - Consistency Around a Cube

Definition of integrability: multidimensional consistency.
(This corresponds to the hierarchy of commuting flows.)

Adjoin a third direction $x_{n,m} \to x_{n,m,k}$ and construct a cube.

$$x_{n,m,k+1} = \bar{x}$$

$$x_{n,m+1,k} = \hat{x}$$

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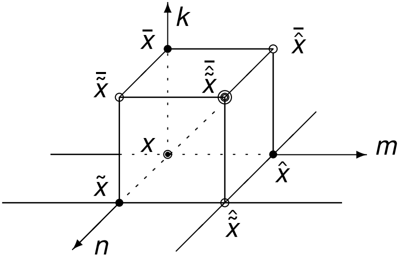

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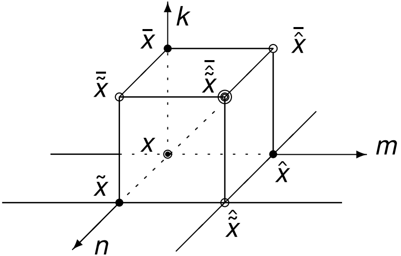

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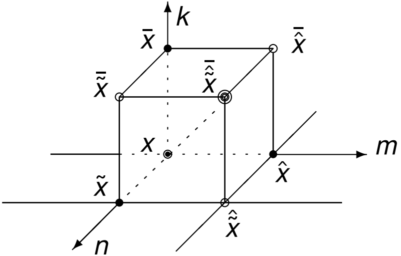
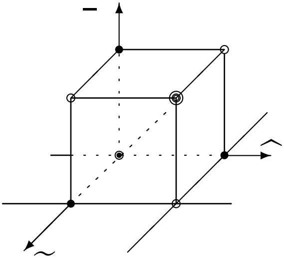


Use *the same map* (with different parameters) also in the (n, k) and (m, k) planes. Identical map on any parallel-shifted plane.

If x, $\tilde{x}$, $\hat{x}$, $\bar{x}$ are given, can solve for $\hat{\tilde{x}}$, $\bar{\tilde{x}}$, $\bar{\hat{x}}$, uniquely. But $\bar{\hat{\tilde{x}}}$ can be computed in 3 different ways and they must agree!

Consistency of pdKdV

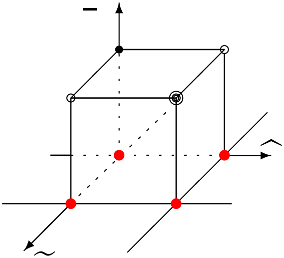
As an example consider the pdKdV equation



$$(x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p = 0,$$

Consistency of pdKdV

As an example consider the pdKdV equation



$$bottom : (x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p \quad = \quad 0,$$

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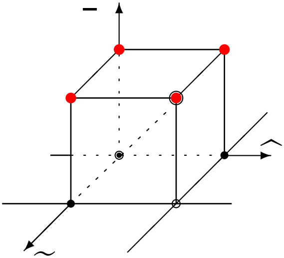

$$\begin{aligned} \mathit{bottom} : (x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p &= 0, \\ \mathit{top} : (\bar{x} - \bar{\hat{\tilde{x}}})(\bar{\tilde{x}} - \bar{\hat{x}}) + q - p &= 0, \end{aligned}$$

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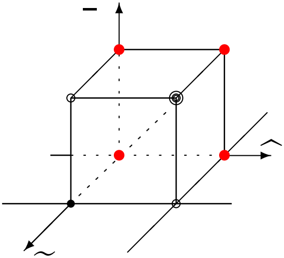

$$
\begin{aligned}
\textit{bottom} : (x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p &= 0,\\
\textit{top} : (\bar{x}-\bar{\hat{\tilde{x}}})(\bar{\tilde{x}}-\bar{\hat{x}})+q-p &= 0,\\
\textit{back} : (x-\hat{\bar{x}})(\bar{x}-\hat{x})+q-r &= 0,
\end{aligned}
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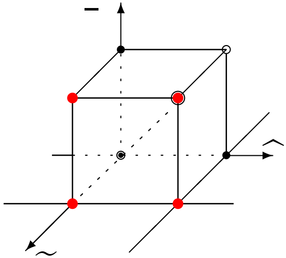

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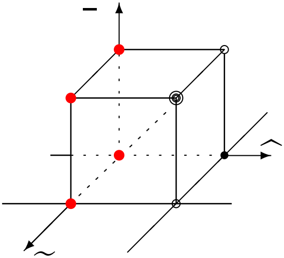

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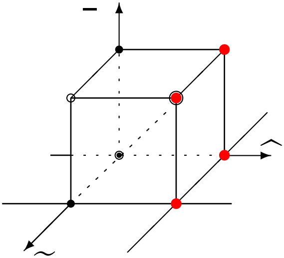

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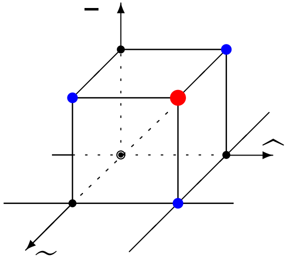

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$$

Solve first for the blue variables, then remaining eqs. all yield

$$
\bar{\hat{\tilde{x}}} = \frac{\tilde{x}\hat{x}(p-q) + \hat{x}\bar{x}(q-r) + \bar{x}\tilde{x}(r-p)}{\tilde{x}(r-q) + \hat{x}(p-r) + \bar{x}(q-p)}
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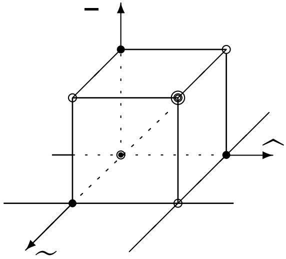

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Note the *tetrahedron property*: the is no unshifted x.

Classification

Adler Bobenko Suris (2003): classification by CAC, but also required <u>exchange symmetry</u> and the <u>tetrahedron property</u>

(H1) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p=0,$$

(H2) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+(q-p)(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})+q^2-p^2=0,$$

(H3) $$p(x\tilde{x}+\hat{x}\hat{\tilde{x}})-q(x\hat{x}+\tilde{x}\hat{\tilde{x}})+\delta(p^2-q^2)=0.$$

(Q1) $$p(x-\widehat{x})(\widetilde{x}-\widehat{\widetilde{x}})-q(x-\widetilde{x})(\widehat{x}-\widehat{\widetilde{x}})=\delta^2 pq(q-p)$$

(Q2) $$p(x-\widehat{x})(\widetilde{x}-\widehat{\widetilde{x}})-q(x-\widetilde{x})(\widehat{x}-\widehat{\widetilde{x}})+pq(p-q)(x+\widetilde{x}+\widehat{x}+\widehat{\widetilde{x}})= pq(p-q)(p^2-pq+q^2)$$

(Q3) $$p(1-q^2)(x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}})-q(1-p^2)(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})= (p^2-q^2)\left((\widehat{x}\widetilde{x}+x\widehat{\widetilde{x}})+\delta^2\frac{(1-p^2)(1-q^2)}{4pq}\right)$$

(Q4) $$\mathrm{sn}(\alpha)(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})-\mathrm{sn}(\beta)(x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}})-\mathrm{sn}(\alpha-\beta)(\widetilde{x}\widehat{x}+x\widehat{\widetilde{x}})+ k\,\mathrm{sn}(\alpha)\mathrm{sn}(\beta)\mathrm{sn}(\alpha-\beta)(1+x\widetilde{x}\widehat{x}\widehat{\widetilde{x}})=0.$$ (JH 2005)

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- CAC but different equations on different sides (e.g. Boll 2010,2011)

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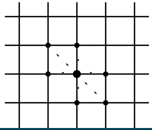
- On a bigger stencil, typical for bilinear equations



$$f_{n+1,m+1}f_{n-1,m-1}o_3 + f_{n+1,m}f_{n-1,m}(o_2 - o_3)$$
$$+f_{n,m+1}f_{n,m-1}(o_1 - o_3) + f_{n,m}^2(o_3 - o_1 - o_2) = 0.$$

Application: Integrable numerical algorithms

Integrable discrete systems often provide good numerical algorithms, because of their inbuilt conservation properties.

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For example:

- Shanks-Wynn's ε acceleration algorithm is related to the discrete potential KdV equation,
- Bauer's η-algorithm is related to the discrete KdV equation.
- Rutishauser's 1954 qd-algorithm for matrix eigenvalues is the discrete Toda lattice equation.
- Recently many algorithms for matrix eigenvalue problems have been related to hungry Lotka-Volterra lattices by Y. Nakamura's group.

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The most important equation is the Hirota-Miwa equation:

$$\boxed{a\,\tilde\tau\,\hat{\bar\tau} + b\,\hat\tau\,\tilde{\bar\tau} + c\,\bar\tau\,\hat{\tilde\tau} + d\,\tau\,\hat{\tilde{\bar\tau}} = 0.}$$

Hirota: $a+b+c=0, d=0$ (KP), Miwa: $a+b+c+d=0$ (BKP)

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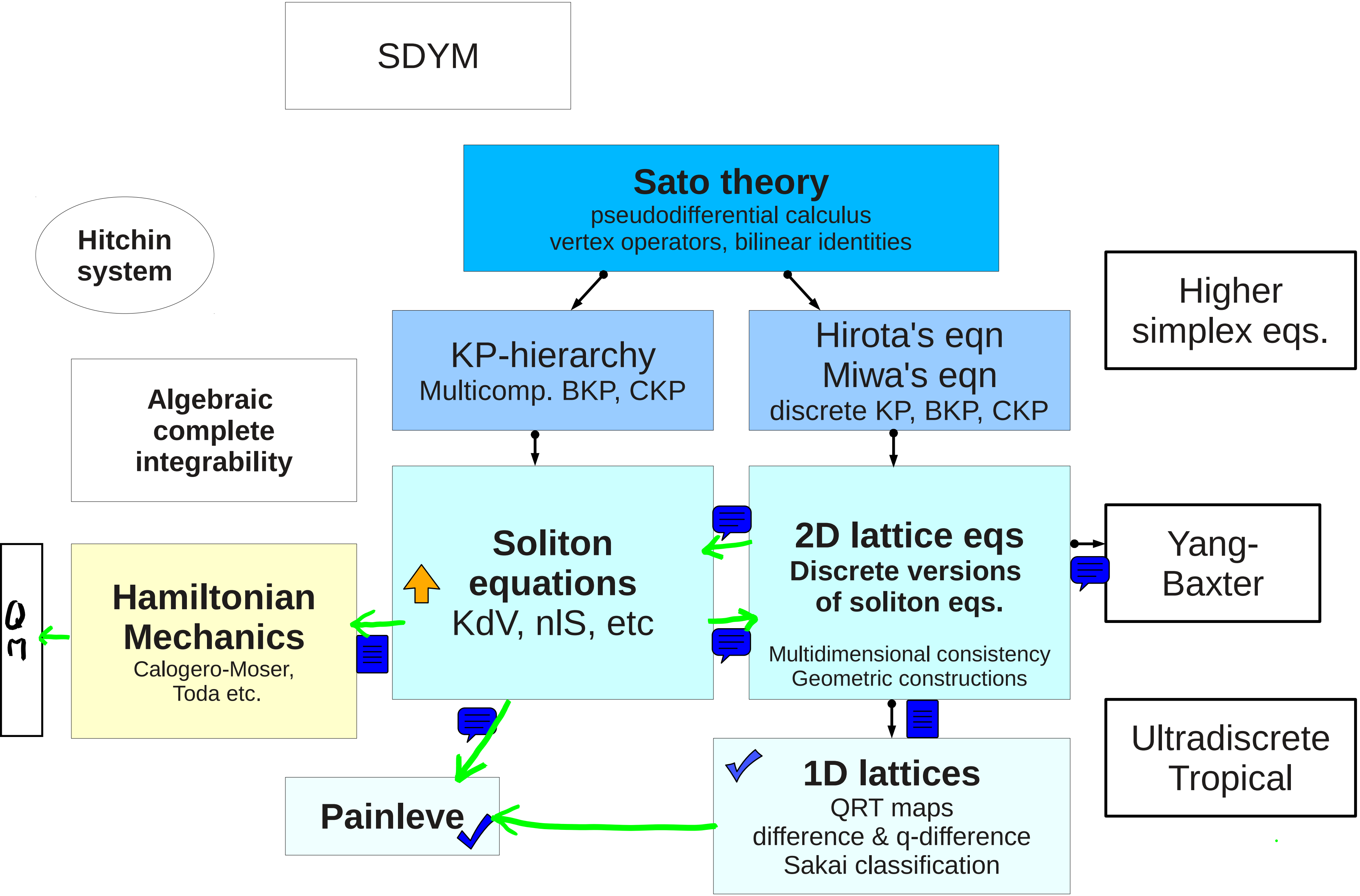
- Recently many classical geometric constructions have been shown to lead to integrable lattice equations (Bobenko, Doliwa, Konopelchenko, Santini, Schief, Suris...)

  These include the theorems of Menelaus and Desargues, the Pentagram map etc.

Overview


SDYM
Sato theory
pseudodifferential calculus
vertex operators, bilinear identities
Hitchin system
KP-hierarchy
Multicomp. BKP, CKP
Hirota's eqn
Miwa's eqn
discrete KP, BKP, CKP
Higher simplex eqs.
Algebraic complete integrability
Soliton equations
KdV, nlS, etc
2D lattice eqs
Discrete versions of soliton eqs.
Multidimensional consistency
Geometric constructions
Yang-Baxter
QM
Hamiltonian Mechanics
Calogero-Moser,
Toda etc.
Painleve
1D lattices
QRT maps
difference & q-difference
Sakai classification
Ultradiscrete
Tropical

Conclusions

- Integrable dynamical equations are interesting in their own right but they are also found in many applications.
- Integrability manifests itself as regular behavior due to conservation laws or other nice mathematical properties.
- The theory of integrable PDEs is well established and a lot is known about the details, as well the general hierarchical structures
- The theory of integrable PΔEs is now under active development and many new things are still to be found.

Conference series

Symmetries and Integrability of Difference Equations

`http://www.side-conferences.net`