Quantum critical phenomena in one dimension

Xi-Wen Guan

中国科学院武汉物理与数学研究所
wuhan Institute of Physics and Mathematics ,Chinese Academy of Sciences

THE AUSTRALIAN NATIONAL UNIVERSITY

In collaboration with M. T. Batchelor and J. Lee

S. Chen, C. Lee, E. Zhao, W. Vincent Liu, M. Oshikawa, T.-L. Ho

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Review

Fermi gases of ultracold atoms in one dimension

I Gaudin-Yang model - Bethe ansatz solutions

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- **solutions to the Fredholm equations**
- **highly polarized fermions - fermionic polarons**
- **highly excited Femionic super Tonks-Girardeau phase**
- **Quantum phases and phase transitions**
- **Luttinger liquids and quantum criticality**
- **spin-charge separation at quantum criticality**
- **Wilson ratio**

II Multicomponent Fermi gases of ultracold atoms

- pairing, trions and universal thermodynamics
- spin-3/2 fermions with $SO(5)$ symmetry
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- universal thermodynamics of $SU(\kappa)$-invariance Fermi gas
- Fermi and Bose mixtures in 1D

III Conformal field theory and correlation functions

- FFLO pairing correlations
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IV Experimental progress

V Concluding remarks

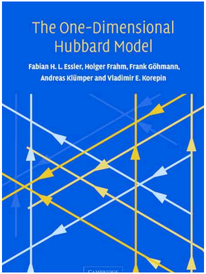
I. Polaron and Molecule states

II. Quantum criticality

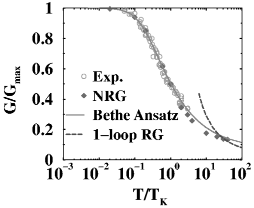
III. FFLO pairing correlation

IV. High spin symmetry

Yang-Baxter solvable models in physics



1D quantum physics



Kondo physics

Exact Solution of the BCS Model

$$H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

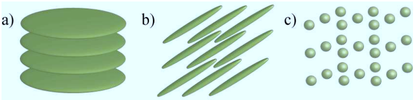
Eigenvalue equation:

$$H_P |\Psi\rangle = E|\Psi\rangle$$

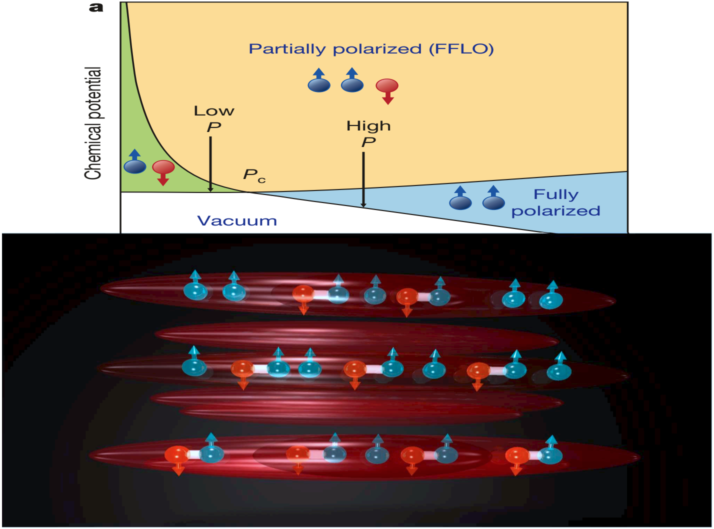
Ansatz for the eigenstates (generalized Cooper ansatz)

$$|\Psi\rangle = \prod_{\alpha=1}^{M} \Gamma^\dagger_\alpha |0\rangle, \quad \Gamma^\dagger_\alpha = \sum_k \frac{1}{2\varepsilon_k - E_\alpha} c^+_{k\uparrow} c^+_{-k\downarrow}$$

Bethe ansatz for ultrasmall metallic grains



Cold atoms in 1D



Experimental measurement of phase diagram of two-component ultracold ^{6}Li atoms trapped in an array of 1D tubes is in excellent agreement with theoretical predictions. Liao *et al*, Nature **467**, 567 (2010)

I. Polaron and Molecule states

Yang-Gaudin model

$$\begin{aligned}\mathcal{H} &= \sum_{j=\downarrow,\uparrow}\int_0^L \phi_j^\dagger(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\phi_j(x)dx \\ &+g_{1D}\int_0^L \phi_\downarrow^\dagger(x)\phi_\uparrow^\dagger(x)\phi_\uparrow(x)\phi_\downarrow(x)dx \\ &-\frac{H}{2}\int_0^L\left(\phi_\uparrow^\dagger(x)\phi_\uparrow(x)-\phi_\downarrow^\dagger(x)\phi_\downarrow(x)\right)dx\end{aligned}$$

- H: effective magnetic field
- $g_{1\mathrm{D}} = -\frac{\hbar^2 c}{m}$, $c = -2/a_{1\mathrm{D}},\ a_{1\mathrm{D}} = -\frac{a_\perp^2}{a_{3\mathrm{D}}} + Aa_\perp$

C. N. Yang,1967; M. Gaudin, 1967
Olshanii M., Phys. Rev. Lett., 81, 938 (1998)

- Bethe Ansatz equations

$$E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2, \;\; \exp(\mathrm{i}k_j L) = \prod_{\ell=1}^{M} \frac{k_j - \Lambda_\ell + \mathrm{i}\,c/2}{k_j - \Lambda_\ell - \mathrm{i}\,c/2},$$

$$\prod_{\ell=1}^{N} \frac{\Lambda_\alpha - k_\ell + \mathrm{i}\,c/2}{\Lambda_\alpha - k_\ell - \mathrm{i}\,c/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}\,c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}\,c}$$

Solutions to the Fredholm equations

- repulsive interaction

$$
\begin{aligned}
r_1(k) &= \frac{1}{2\pi} + \int_{-B_2}^{B_2} K_1(k-k')r_2(k')dk', \\
r_2(k) &= \int_{-B_1}^{B_1} K_1(k-k')r_1(k')dk - \int_{-B_2}^{B_2} K_2(k-k')r_2(k')dk' \\
n: &\equiv N/L = \int_{-B_1}^{B_1} r_1(k)dk, \;\; n_\downarrow :\equiv N_\downarrow/L = \int_{-B_2}^{B_2} r_2(k)dk, \\
E &= \int_{-B_1}^{B_1} k^2 r_1(k)dk, \;\; h = 2\frac{\partial E(n,s_z)}{\partial s_z}, \qquad \mu = \frac{\partial E(n,s_z)}{\partial n}
\end{aligned}
$$

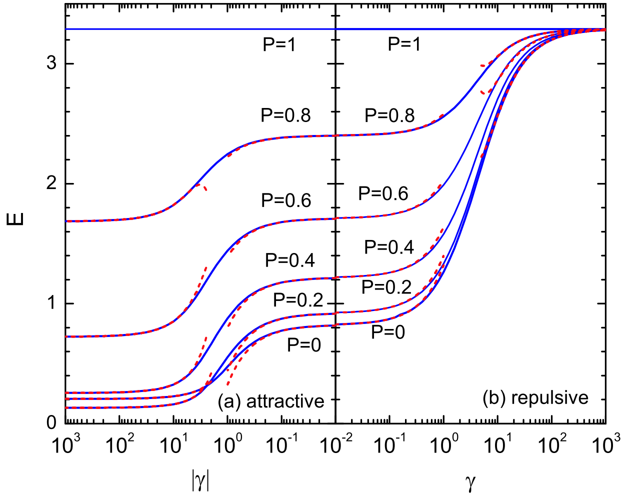
- attractive interaction

$$\rho_1(k) = \frac{1}{2\pi} + \int_{-Q_2}^{Q_2} K_1(k-k')\rho_2(k')dk'$$

$$\rho_2(k) = \frac{2}{2\pi} + \int_{-Q_1}^{Q_1} K_1(k-k')\rho_1(k')dk' + \int_{-Q_2}^{Q_2} K_2(k-k')\rho_2(k')dk'$$

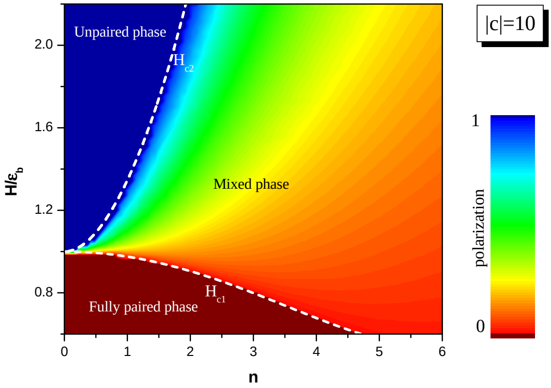
$$n \equiv: \frac{N}{L} = 2\int_{-Q_2}^{Q_2} \rho_2(k)dk + \int_{-Q_1}^{Q_1} \rho_1(k)dk, \quad n_\downarrow \equiv: \frac{N_\downarrow}{L} = \int_{-Q_2}^{Q_2} \rho_2(k)dk$$

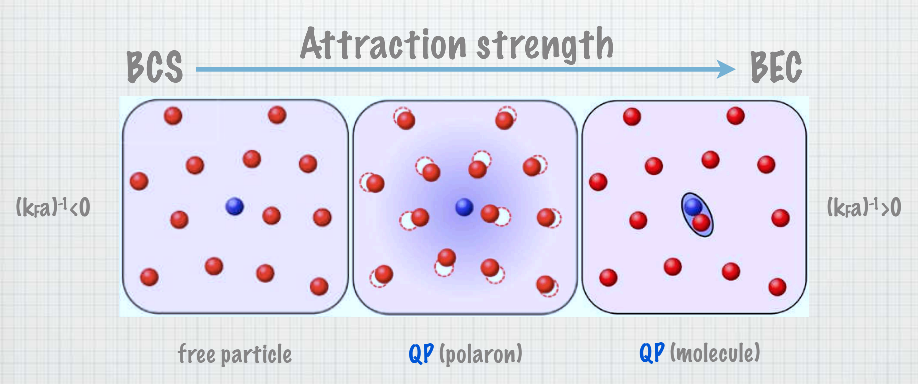
$$E = \int_{-Q_2}^{Q_2} \left(2k^2 - c^2/2\right)\rho_2(k)dk + \int_{-Q_1}^{Q_1} k^2\rho_1(k)dk$$



Guan, Batchelor, Lee & Bortz, Phys. Rev. B (2007)
Guan & Ma, Phys. Rev. A (2012)


BCS
h_{c1}
FFLO
h_{c2}
Normal
h


Unpaired phase
H_{c2}
Mixed phase
H_{c1}
Fully paired phase
|c|=10
H/ε_b
n
polarization
2.0
1.6
1.2
0.8
0
1
2
3
4
5
6
1
0

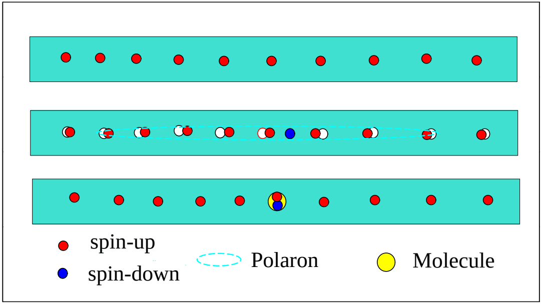


Fermi polaron is a dressed spin-down impurity fermion by the surrounding scattered fermions in a spin-up Fermi sea. With increasing attraction, the single spin-down fermion possibly undergoes a polaron-molecule transition in the Fermionic medium.

Zwierlein's group, PRL (2009)

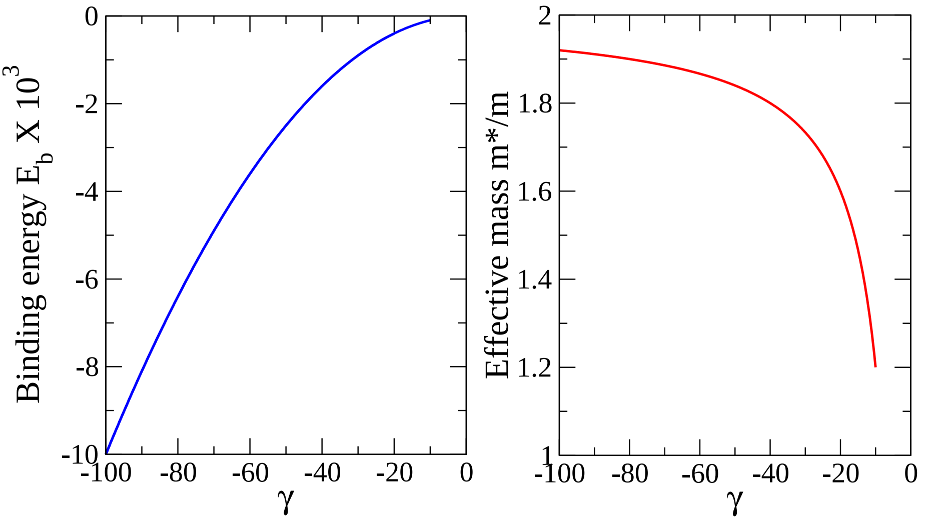
Kohstall *et al*, Nature 2012; Koschorreck *et al*, Nature 2012

highly polarized fermions - fermionic polarons



$$E_p(p) = E_b + \frac{\hbar^2 p^2}{2m^*}$$

McGuire, J. Math. Phys. **6**, 432 (1965); **7**, 123 (1966)
Gu & Yang, Commun. Math. Phys. **122**, 105 (1989)
Guan, Frontiers in Physics, **7**, 8 (2012)



Polaron $m^* \approx m(1+O(c^2)), \qquad E_b \approx -\frac{6}{\pi^2}\varepsilon_F|\gamma|, \qquad \gamma \ll 1$

Molecule $m^* \approx 2m\left(1-\frac{4}{|\gamma|}\right), \qquad E_b \approx \frac{\hbar^2 n^2}{2m}\left(-\frac{\gamma^2}{2}+\frac{8\pi^2}{3|\gamma|}\right), \qquad \gamma \gg 1$

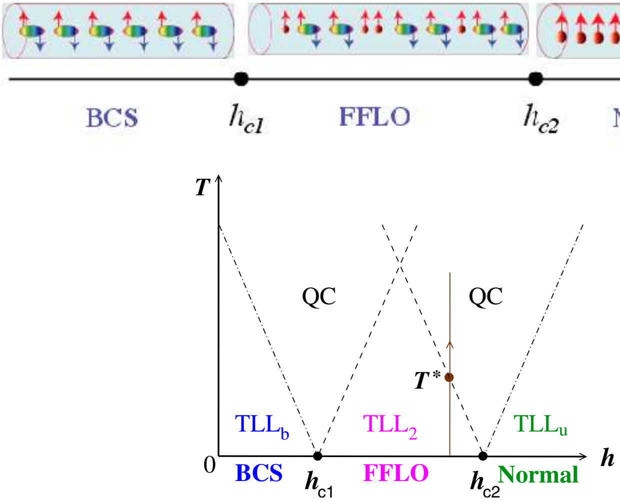
II. Quantum criticality

$$
\begin{aligned}
\sigma(k) + \sigma^h(k) &= \frac{1}{\pi} - a_2 * \sigma(k) - a_1 * \rho(k) \\
\rho(k) + \rho^h(k) &= \frac{1}{2\pi} - a_1 * \sigma(k) - \sum_{n=1}^{\infty} a_n * \xi_n(k) \\
\xi_n(\Lambda) + \xi_n^h(\Lambda) &= a_n * \rho(\Lambda) - \sum_{n=1}^{\infty} T_{nm} * \xi_n(\Lambda) \\
(f * g)(\lambda) &= \int_{-\infty}^{\infty} f(\lambda - \lambda')g(\lambda')d\lambda', \;\; a_m(\lambda) = \frac{1}{2\pi} \frac{m|c|}{(mc/2)^2 + \lambda^2}
\end{aligned}
$$

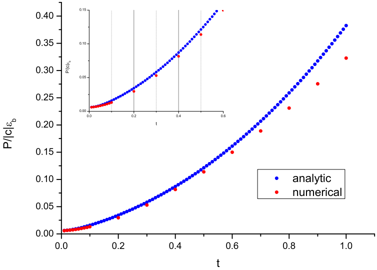
$$
\begin{aligned}
Z &= \mathrm{tr} e^{-\frac{\mathcal{H}}{T}} = \sum_{\sigma, \sigma^h, \rho, \rho^h, \xi_n, \xi_n^h} W e^{-\frac{E(\sigma, \sigma^h, \rho, \rho^h, \xi_n, \xi_n^h)}{T}} \\
S &= \ln W, \;\; G = E - \mu N - HM^z - TS
\end{aligned}
$$

Takahashi, Thermodynamics of One-Dimensional Solvable Models

The decision speed and accuracy often increases with the number of decision makers. Although their individual motions are complex, their collective behaviour acquires qualitatively new forms of simplicity - collective motion of “particles”



Guan, Batchelor, Lee & Bortz, Phys. Rev. B (2007)
Zhao, Guan, Liu, Batchelor & Oshikawa, Phys. Rev. Lett. (2009)

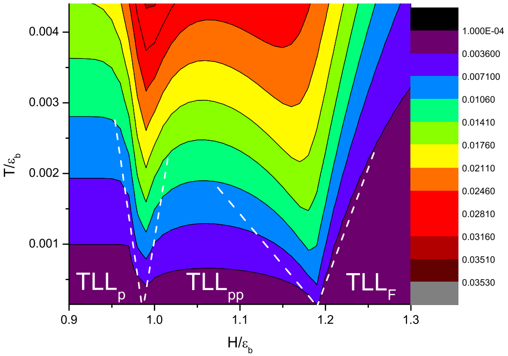


Equation of State: $\tilde{p} = \tilde{p}^b + \tilde{p}^u$

$$\tilde{p}^b = -\frac{t^{3/2} f^b_{3/2}}{2\sqrt{\pi}} \left(1 - \frac{t^{3/2} f^b_{3/2}}{16\sqrt{\pi}} - \frac{t^{3/2} f^u_{3/2}}{\sqrt{2\pi}} \right)$$

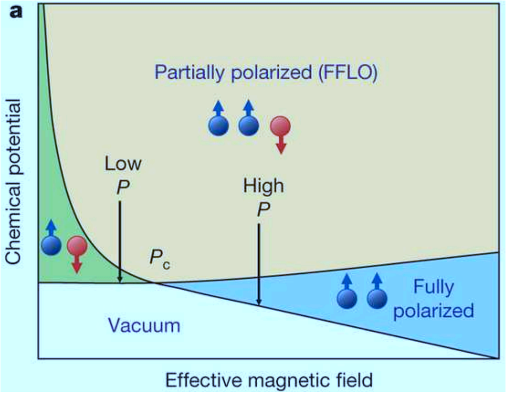
$$\tilde{p}^u = -\frac{t^{3/2} f^u_{3/2}}{2\sqrt{2\pi}} \left(1 - \frac{t^{3/2} f^b_{3/2}}{\sqrt{\pi}} \right) + f_{\mathrm{spin-wave}}$$

$$f^b_n = \mathrm{Li}_n\left(-e^{A_b/t}\right), \; f^u_n = \mathrm{Li}_n\left(-e^{A_u/t}\right), \; \mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$



$$TLL_p:\ S = \frac{\pi k_B T}{3\hbar}\frac{1}{v_{\mathrm{b}}},\quad TLL_F:\ S = \frac{\pi k_B T}{3\hbar}\frac{1}{v_F}$$

$$TLL_{pp}:\ S = \frac{\pi K_B T}{3\hbar}\left(\frac{1}{v_{\mathrm{b}}}+\frac{1}{v_{\mathrm{u}}}\right)$$

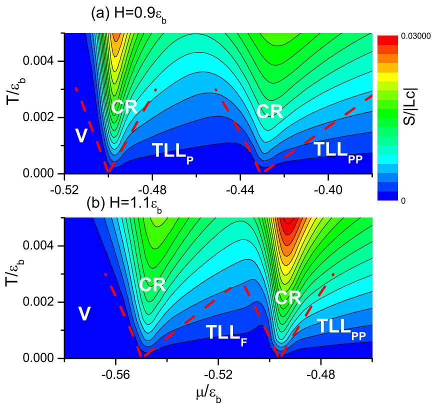


$$(V{-}F): \ \tilde{\mu}_{c1} = -\frac{h}{2}, \quad (V{-}P): \ \tilde{\mu}_{c2} = -\frac{1}{2}$$

$$(F{-}PP): \ \tilde{\mu}_{c3} = -\frac{1}{2}\left(1 - \frac{2}{3\pi}(h-1)^{\frac{3}{2}} - \frac{2}{3\pi^2}(h-1)^2\right)$$

$$(P{-}PP): \ \tilde{\mu}_{c4} = -\frac{h}{2} + \frac{4}{3\pi}(1-h)^{\frac{3}{2}} + \frac{3}{2\pi^2}(1-h)^2$$

Orso, Phys. Rev. Lett. (2007); Hu, Liu & Drummond, Phys. Rev. Lett. (2007)
Guan, Batchelor, Lee & Bortz, PRB (2007)

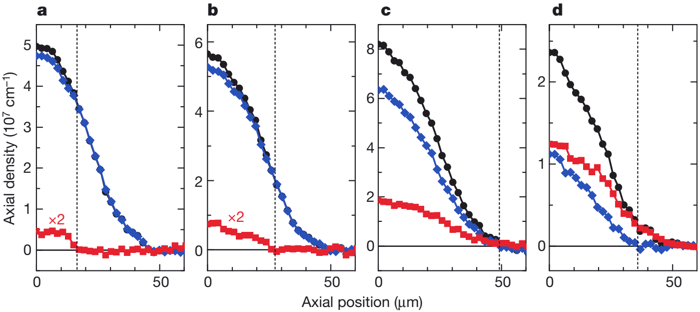


$$n(\mu, T) = n_0 + T^{\frac{d}{z}+1-\frac{1}{\nu z}} \mathcal{G}\left(\frac{\mu-\mu_c}{T^{\frac{1}{\nu z}}}\right), \quad \Delta \sim |\mu-\mu_c|^{z\nu}, \quad z = 2$$

$$\kappa(\mu, T) = \kappa_0 + T^{\frac{d}{z}+1-\frac{2}{\nu z}} \mathcal{F}\left(\frac{\mu-\mu_c}{T^{\frac{1}{\nu z}}}\right), \quad \xi \sim |\mu-\mu_c|^{-\nu}, \quad \nu = 1/2$$

Guan & Ho, 2011; Guan & Batchelor, 2011

Fisher *et al*, 1989; Zhou& Ho 2010

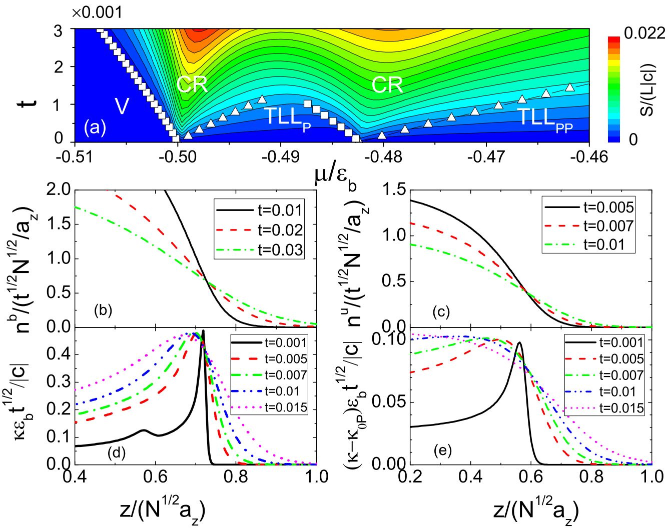


Local Density Approximation $\mu_{\text{hom}}(n(x), P(x)) = \mu_0 - \frac{1}{2}m\omega_x^2 x^2$
($P = 0.015, 0.055, 0.1, 0.33$)

$$Na_{1D}^2/a_x^2 = 4\int_{-\infty}^{\infty} \tilde{n}(x)\, d\tilde{x}$$

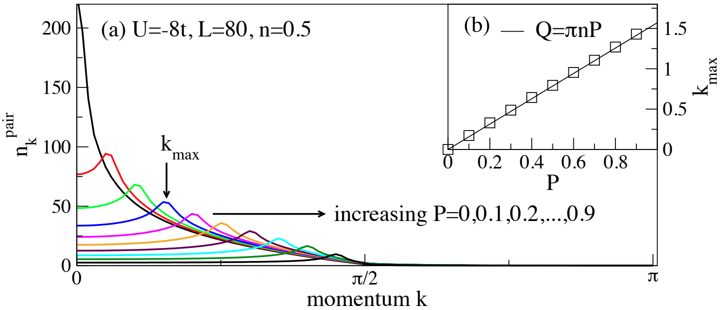
$$\left(Na_{1D}^2\right) P = 4\int_{-\infty}^{\infty} \tilde{n}^u(x)\, d\tilde{x} \times a_x^2$$

Ma & Yang, Chinese Phys. Lett. (2010),(2011)
Orso, Phys. Rev. Lett. (2007)

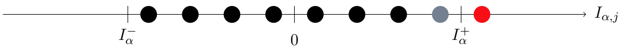


Yin, Guan, Batchelor & Chen, PRA 2011

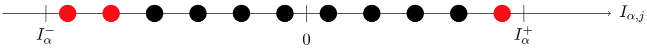
III. FFLO pairing correlations



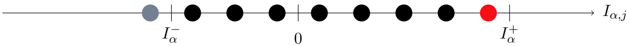
- **1D attractive Hubbard model (Bogoliubov and Korepin, 1988, 1889): the single particle Green's function** $\langle \psi^\dagger_{n,s}\psi_{1,s}\rangle \to e^{-n/\xi},\ \xi = v_F/\Delta$ **the singlet pair correlation function** $\langle \psi^\dagger_{n,\uparrow}\psi^\dagger_{n,\downarrow}\psi_{1,\uparrow}\psi_{1,\downarrow}\rangle \to n^{-\theta}$
- **Fulde & Ferrell: Cooper pairs form with finite centre-of-mass momentum.**
- **Larkin & Ovchinnikov: the order parameter oscillates in space.**
- **the 1D FFLO state (Feiguin & Heidrich-Meisner, 2007; M. Tezuka & M. Ueda 2008): numerical evidence of the power-law decay** $n^{\rm pair} \propto \cos(k_{\rm FFLO}|x|)/|x|^{\alpha}$



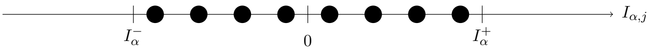
particle-hole excitation



adding particles



backscattering process



ground state

Low energy excitations and conformal dimensions

conformal dimensions

$$2\Delta_u^{\pm} = \left(Z_{uu}\Delta D_u + Z_{bu}\Delta D_b \pm \frac{Z_{bb}\Delta N_u - Z_{ub}\Delta N_b}{2\det Z}\right)^2 + 2N_u^{\pm},$$

$$2\Delta_b^{\pm} = \left(Z_{ub}\Delta D_u + Z_{bb}\Delta D_b \pm \frac{Z_{uu}\Delta N_b - Z_{bu}\Delta N_u}{2\det Z}\right)^2 + 2N_b^{\pm},$$

two-point correlation functions at $T = 0$

$$\langle O(x,t)O(0,0)\rangle = \frac{\exp(-2\mathrm{i}(N_u\Delta D_u + N_b\Delta D_b)x)}{(x - \mathrm{i}v_u t)^{2\Delta_u^+}(x + \mathrm{i}v_u t)^{2\Delta_u^-}(x - \mathrm{i}v_b t)^{2\Delta_b^+}(x + \mathrm{i}v_b t)^{2\Delta_b^-}}$$

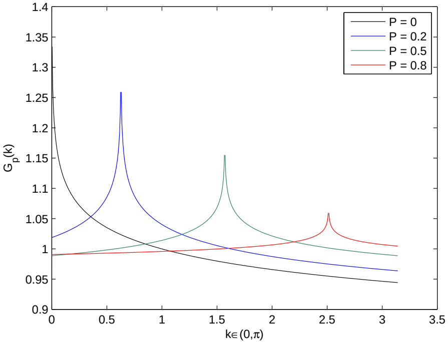
$$\Delta D_u \equiv \frac{\Delta N_u + \Delta N_b}{2} \pmod 1, \qquad \Delta D_b \equiv \frac{\Delta N_u}{2} \pmod 1$$

Essler, Frahm, Göhmann, Klümper & Korepin, *The One-Dimensional Hubbard Model*, (2005).

Pair correlation function

$$\langle \psi_\uparrow^\dagger(x,t)\psi_\downarrow^\dagger(x,t)\psi_\uparrow(0,0)\psi_\downarrow(0,0)\rangle \approx \frac{A_{p,1}\cos\left(\pi(n_\uparrow - n_\downarrow)x\right)}{|x+\mathrm{i}v_u t|^{\theta_1}|x+\mathrm{i}v_b t|^{\theta_2}}$$

$$\theta_1 \approx \frac{1}{2}, \qquad \theta_2 \approx \frac{1}{2} + \frac{(1-P)}{2|\gamma|}$$



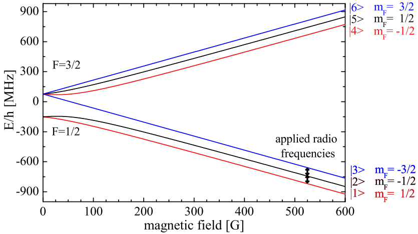
The spatial modulations are characteristic of a Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state. The backscattering among the Fermi points of bound pairs and unpaired fermions results in a 1D analog of the FFLO state and displays a microscopic origin of the FFLO nature.

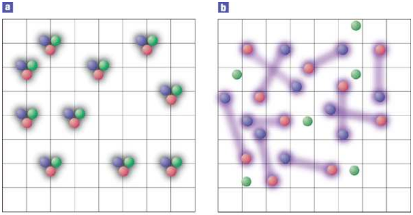
Lee & Guan, Nucl. Phys. B (2011)

Schlottmann and Zvyagin, **85**, (2012)

Although high symmetries do not occur frequently in nature, they deserve attention since every new symmetry brings with itself a possibility of new physics.
(Controzzi, and Tsvelik, PRL, 2006)

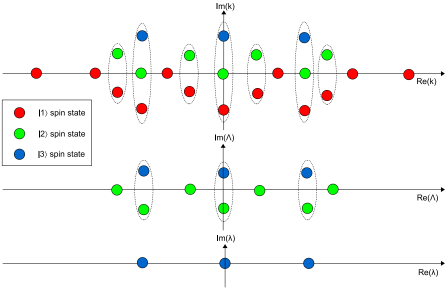
IV. Multicomponent Fermi gases of ultracold atoms





- **Fermionic atoms:** ^{6}Li I=1, S=1/2; ^{40}K I=4, S=1/2, e.g. the two-component atomic mixture is created in the lowest two states of ^{6}Li atoms (Grimm et al, 07).
- Realization of a $SU(2) \times SU(6)$ system of fermions with ytterbium ^{171}Yb ($I = 1/2$) and ^{173}Yb ($I = 5/2$) atoms (Taie *et al*, PRL 2010). $SU(6)$ symmetric Mott-insulator state with ^{173}Yb atoms (Taie *et al*, Nature Phys. 2012).
- Atomic Fermi gases with multi-component hyperfine states are tunable interacting many-body systems featuring novel and subtle quantum phase transitions. (Ho, Yi, PRL (1999); Wilczek(Nature Physics).

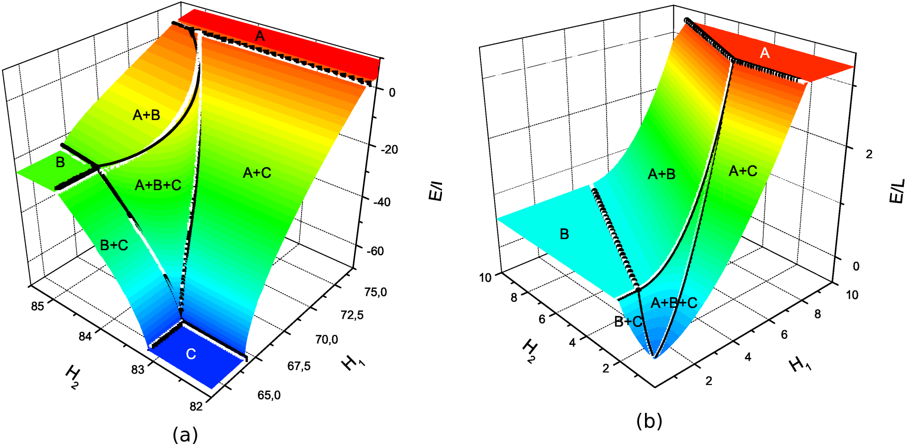
Pairing, trions and universal thermodynamics



$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N_f}\frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}}\sum_{1\le i<j\le N_f}\delta(x_i - x_j) + E_z$$

$$E_0 \approx \sum_{r=1}^{3}\frac{\pi^2 n_r^3}{3r}\left(1 + \frac{2}{|c|}A_r + \frac{3}{c^2}A_r^2\right) - \sum_{r=2}^{N} n_r \epsilon_r$$

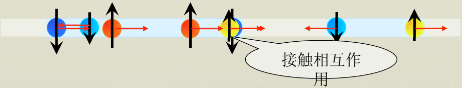
Guan, Batchelor, Lee, Zhou, Phys. Rev. Lett. (2008)
Guan, Lee, Batchelor, Yin, Chen, Rev. A (2010)
Kuhn and Forster, New. J. Phys. (2012)



$H_1 = 2c^2/3 + (\mu^{\mathrm{u}} - \mu^{\mathrm{t}}),\; H_2 = 5c^2/6 + 2(\mu^{\mathrm{b}} - \mu^{\mathrm{t}}),\; H_1 - H_2/2 = c^2/4 + (\mu^{\mathrm{u}} - \mu^{\mathrm{b}})$

Integrable spin-3/2 fermions with $SO(5)$ and $SO(4)$ symmetries

自旋-3/2费米子， Hamiltonian的对称性



$$\hat{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}^{N} \sum_{lm} g_{lm} \hat{P}_{ij}^{lm} \delta(x_i - x_j) - h\hat{M}.$$

No	$\|0,0\rangle$	$\|2,2\rangle$	$\|2,1\rangle$	$\|2,0\rangle$	$\|2,-1\rangle$	$\|2,-2\rangle$	对称性	可积性
·1	c	c	c	c	c	c	$SU(4)$	√
2	c_0	c_2	c_2	c_2	c_2	c_2	$SO(5)$	$c_0=c_2$
3	c_a	c_b	c_a	c_b	c_a	c_b	$SO(4)$	$c_a=-c_b$

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{\ell<j}^{N} \left(c_0 + c_2 \mathbf{S}_j \mathbf{S}_\ell\right) \delta(x_j - x_\ell)$$

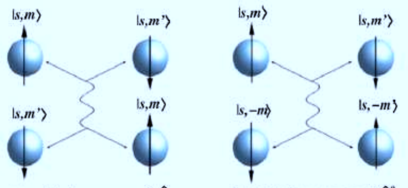
$$S_{jl} = \frac{k_j - k_l - \mathrm{i}\frac{3c}{2}}{k_j - k_l + \mathrm{i}\frac{3c}{2}} P_{jl}^0 + P_{jl}^1 + \frac{k_j - k_l - \mathrm{i}\frac{c}{2}}{k_j - k_l + \mathrm{i}\frac{c}{2}} P_{jl}^2 + P_{jl}^3$$

Jiang, Cao & Wang, EPL (2009)

Integrable spin-3/2 fermions with $SO(4)$ symmetry

费米玻色SU(2s+1） $\hat{N}_m = \sum_i \hat{\psi}^+_{i,m}\hat{\psi}^-_{i,m}, m = -s,\cdots,s+1,s.$

费米 Sp(2s+1)， SO(4)
玻色 SO(2s+1) $J_m = \sum_i (\hat{\psi}^+_{i,m}\hat{\psi}^-_{i,m} - \hat{\psi}^+_{i,-m}\hat{\psi}^-_{i,-m}), m = -s,\cdots,s+1,s.$

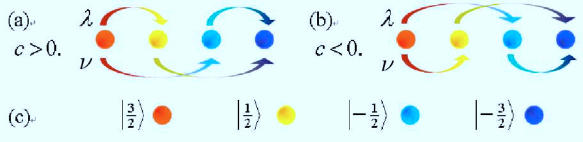


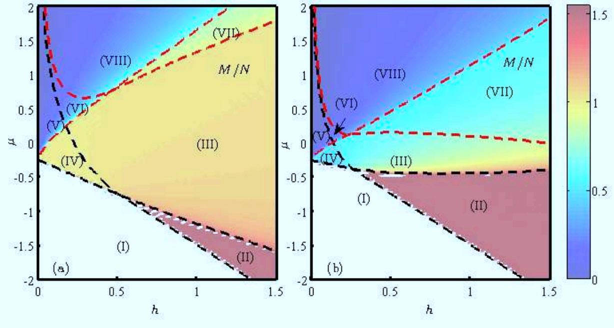
(a) scattering process of $\hat{P}_{ab}$ (b) scattering process of $\hat{P}^0_{ab}$

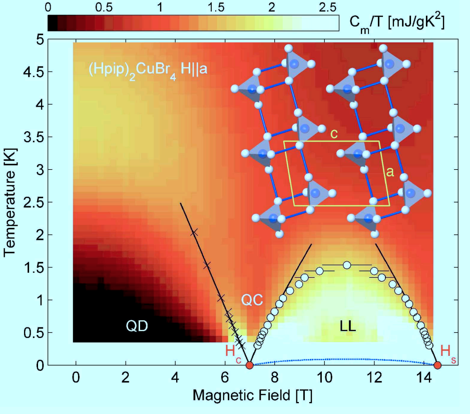
$$\hat{H} = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \sum_{i\neq j}^{N}\sum_{lm} g_{lm}\hat{P}^{lm}_{ij}\delta(x_i - x_j) - h\hat{M}$$

$$S_{ab}(k) = \frac{k+\mathrm{i}c}{k-\mathrm{i}c}V_{1,ab} + \frac{k-\mathrm{i}c}{k+\mathrm{i}c}V_{2,ab} + \sum_{l=1,3}\sum_{m} P^{lm}_{ab}$$

Integrable spin-3/2 fermions with $SO(4)$ symmetry

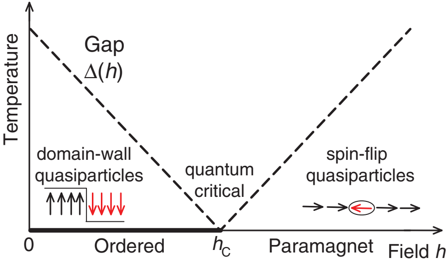


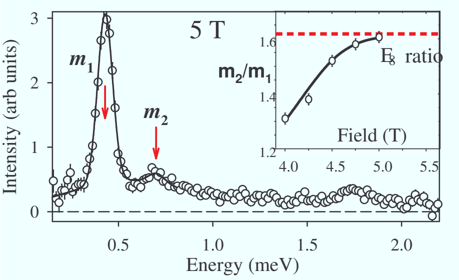




Field-emperature phase diagram of the strong coupling spin-ladder compound $(Hpip)_2CuBr_4$ with $J_\mathrm{r} = 12.9$K, $J_\mathrm{l} = 3.3$K (Thielemann *et al*, PRL, 2008; Ruegg *et al.* PRL, 2008). The quantum criticality near the critical point is mapped onto free-Fermi theory with $z = 2$ and the correlation length exponent $\nu = 1/2$ (Maeda *et al*, PRL, 2007). In the LL phase, the critical exponents $z = 1$ and $\nu = 1$.

$$C_m = \frac{\pi T}{3 v_s}, \quad M \sim -\sqrt{\frac{mT}{2\pi}} \mathrm{Li}_{1/2}\left(-e^{(h-\Delta)/T}\right), \; v_s = \sqrt{\Delta/m}$$





The exceptional simple group E_8 not only is important for super-string theory but also can be a true symmetry in a simple Ising magnet. The integrable quantum field theory describes the scaling limit of $T = T_c$ Ising model with both transverse and non-zero magnetic field. The quantum Ising model with transverse field displays E_8 symmetry in excitation spectrum. The ratio of the first two bound state energies as the critical point is approached is close to the golden ratio $m_2/m_1 = (\sqrt{5}+1)/2$. (Zamolodchikov I989; Coldea *et al*, Science 327, (2010) 177)

V. Concluding remarks

1. **Exactly solved models provide precise understanding of critical phenomena in quantum systems.**
2. **Mathematical theory of this kind has now become testable in experiments.**
3. **Mathematical symmetry can reveal true beauty in physics.**
4. **The future is very bright for more breakthroughs in this area.**

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