Asymptotics of spacing distributions in RMT

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Outline

- Historical overview
- The log-gas heuristics
- Some rigorous results
- A conjecture

Spacing distributions 1962— present

Introduction — a numerical experiment

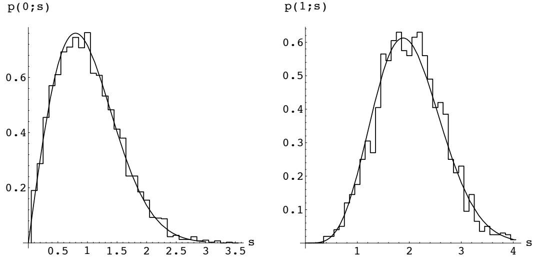
X — an $n \times n$ matrix with random entries from $\mathrm{N}[0, 1]$.

$R := (X + X^T)/2$ — an $n \times n$ real symmetric matrix from the GOE_n.

Let $p^{\mathrm{bulk}}(s; \mathrm{GOE}_n)$ denote the probability density for the distribution of the spacing between eigenvalues $[n/2]$ and $[n/2] + 1$. An approximation can be computed through simulation:

- generate M members of the GOE, compute $\lambda_{[n/2]+1} - \lambda_{[n/2]}$ for each;
- scale the resulting list so that the mean is unity;
- form a histogram.

Wigner surmise



- $M = 2,000$, $n = 13$.
- $p(0;s) = p^{\mathrm{bulk}}(s;\mathrm{GOE}_n)$
- $p(1;s)$ is the distribution of the spacing $\lambda_{[n/2]+2} - \lambda_{[n/2]}$ i.e. bulk second nearest neigbours.
- Solid curves are the Wigner surmises

$$p^{\mathrm{W}}(0;s) = \frac{\pi}{2} s e^{-\pi s^2/4}, \qquad p^{\mathrm{W}}(1;s) = \frac{2^{18} s^4}{3^6 \pi^3} e^{-64 s^2/9\pi}$$

Exact form of $p^{\mathrm{bulk}}(s; \mathrm{GOE}_n)$ for $n \to \infty$

- In 1961 it was shown by Gaudin that $\lim_{n\to\infty} p^{\mathrm{bulk}}(s; \mathrm{GOE}_n) = \frac{d^2}{ds^2} \det(1 - K_{(0,s)})$ where $K_{(0,s)}$ is the integral operator on $(0, s)$ with kernel

$$K(x, y) = \frac{1}{2}\Big(\frac{\sin \pi(x-y)}{\pi(x-y)} + \frac{\sin \pi(x+y)}{\pi(x+y)}\Big).$$

- In 1980 it was shown by the Kyoto school of Jimbo et al that this same Fredholm determinant can be expressed in terms of a solution of a sigma Painlevé V equation.
- Notice that the functional form of $p^{\mathrm{W}}(0; s)$ is $p^{\mathrm{W}}(0; s) = a(s) \exp(-\int_0^s a(t)\, dt)$. Forrester and Witte (2001) showed, that

$$\lim_{n\to\infty} p^{\mathrm{bulk}}(s; \mathrm{GOE}_n) = \frac{2u((\pi s/2)^2)}{s} \exp\Big(-\int_0^{(\pi s/2)^2} \frac{u(t)}{t}\, dt\Big)$$

where, with $u(s) \underset{s\to 0^+}{\sim} \frac{s}{3} - \frac{s^2}{45} + \frac{8s^{5/2}}{135\pi}$,

$$s^2(u'')^2 = (4(u')^2 - u')(su' - u) + \frac{9}{4}(u')^2 - \frac{3}{2}u' + \frac{1}{4}$$

Application of exact spacing distribution

The Montgomery-Odlyzko law states that the statistics of the large Riemann zeros coincide with the statistics of the bulk eigenvalues for GUE matrices — matrices $(X + X^\dagger)/2$ with X an $n \times n$ complex standard Gaussian.

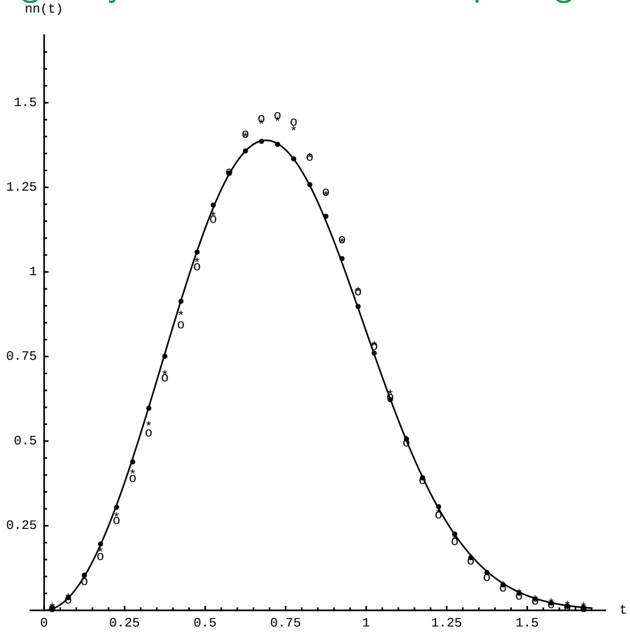
Odlyzko has a generated a famous data set of the Riemann zeros. The first sentence of his 1987 paper "The 10^{20}-th zero of the Riemann zeta function and 70 million of its neighbors" reads

`The` 10^{20}`-th zero of the Riemann zeta function equals`

$$\frac{1}{2} + i\,15202440115920747268.6290299\ldots$$

At this time he also computed 70 million of its neighbours. Such accurate statistics can distinguish the Wigner surmise from the exact result.

Graph using Odlyzko's data and exact spacing distribution



Large s asymptotics

The eigenvalue PDF for the GOE ($\beta = 1$), GUE ($\beta = 2$) and GSE ($\beta = 4$) is proportional to

$$\prod_{j=1}^{N} e^{-\beta\lambda_j^2/2} \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|^\beta$$

Denote by $E_\beta^{\rm bulk}(k;s)$ the probability that after bulk scaling there are k eigenvalues in the interval $(0,s)$. Dyson used a macroscopic log-gas argument to predict that

$$E_\beta^{\rm bulk}(k;s/\pi) \underset{s\to\infty}{\sim} \tau_\beta s^{-(3-\beta/2+2/\beta)} e^{-\beta s^2/16+(\beta/2-1)s/2}.$$

This is verified for $\beta = 1,2$ and 4, and the Fredholm/Painlevé characterisation gives

$$\tau_1 = 2^{5/12} e^{(3/2)\zeta'(-1)}, \quad \tau_2 = 2^{1/3} e^{3\zeta'(-1)}, \quad \tau_4 = 2^{-29/24} e^{(3/2)\zeta'(-1)}.$$

Log-gas strategy, applied to CβE$_N$

First approach (Dyson 1962)

- Introduce the large deviations ansatz

$$E_\beta(0; (-\alpha, \alpha); \mathrm{C}\beta\mathrm{E}_N) \underset{N\to\infty}{\sim} e^{-\beta \delta F}$$

where δF is the free energy cost of conditioning the equilibrium density so that $\rho_{(1)}(\theta) = 0$ for $\theta \in (-\alpha, \alpha)$.

- Choose $\rho_{(1)}(\theta)$ to minimize $\delta F = V_1 + V_2$, where V_1 is the electrostatic energy

$$V_1 = -\frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \big(\rho_{(1)}(\theta_1) - N/2\pi\big)\big(\rho_{(1)}(\theta_2) - N/2\pi\big) \log|e^{i\theta_1} - e^{i\theta_2}|\, |d\theta_1 d$$

and the entropy term

$$V_2 = \left(\frac{1}{\beta} - \frac{1}{2}\right) \int_0^{2\pi} \rho_{(1)}(\theta) \log\left(\frac{\rho_{(1)}(\theta)}{N/2\pi}\right)$$

Up to terms of order $\log(N\alpha)$, suffices to minimise V_1. This gives

$$\rho_{(1)}(\theta) = N \frac{\sin(\theta/2)}{\sqrt{\sin^2(\theta/2) - \sin^2(\alpha/2)}}$$

and

$$\beta V_1 = -\frac{\beta}{2} N^2 \log\cos\left(\frac{\alpha}{2}\right), \quad \beta V_2 = \left(1 - \frac{\beta}{2}\right) N \log\left(\sec\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\alpha}{2}\right)\right)$$

Now take a double scaling limit, replacing $(-\alpha, \alpha)$ by $(-\pi s/N, \pi s/N)$, and then taking $N \to \infty$.

This gives

$$E_\beta^{\mathrm{bulk}}(0;(0,s)) \underset{s\to\infty}{\sim} \exp\Big(-\beta(\pi s^2)/16 + (\beta/2 - 1)\pi s/2\Big).$$

Log-gas strategy applied directly to the bulk

Second approach (Dyson (1995), Fogler and Shklovskii (1995))

- The aim is to compute the asymptotics of $E_\beta^{\rm bulk}(n;s)$.
- The log-gas is taken to be infinite in extent, with the bulk state characterised by a uniform density.
- The n eigenvalues are taken to be a continuous conductive fluid occupying the interval $(-b,b) \subset (-t,t)$, $2t = s$.
- This region has constant electrostatic potential $-v$ say, while the electrostatic potential in $\mathbb{R}\backslash\{-t,t\}$ is zero.
- Find the simple result $V_1 = -\frac{nv}{2} + \frac{\pi^2}{4}(t^2 - v^2)$, $V_2 = v$.
- b is determined by n, and similarly v, in terms of elliptic integrals.

Result for $E_\beta^{\text{bulk}}(n;(0,s))$

This gives that for $0 \ll n \ll s$, we have

$$\log E_\beta^{\text{bulk}}(n;(0,s)) \underset{s\to\infty}{\sim} -\beta\frac{(\pi s)^2}{16} + \left(\beta n + \frac{\beta}{2} - 1\right)\frac{\pi s}{2} + \left\{\frac{n}{2}\left(1 - \frac{\beta}{2} - \frac{\beta n}{2}\right) + \frac{1}{4}\left(\frac{\beta}{2} + \frac{2}{\beta} - 3\right)\right\}\log s$$

Rigorous results

There are characterisations of the bulk general β state in terms of stochastic differential equations due to Killip and Stoiciu, and Valko and Virág.

The latter have used this, and the Cameron-Martin-Girsanov formula to prove the asymptotic formula

$$E_\beta^{\rm bulk}(0;(0,s)) \underset{s\to\infty}{\sim} \exp\Big(-\beta(\pi s^2)/16 + (\beta/2-1)\pi s/2\Big).$$

The results for $\log E_\beta^{\rm bulk}(n;(0,s))$ can be established (and extended) for $\beta = 1, 2$ and 4 using results for the eigenvalues of the underlying Fredholm operator.

My results on the topic

- I've used generalized hypergeometric functions to obtain rigorous results for the asymptotics of $E_\beta^{\mathrm{hard}}(0;(0,s);a)$
- Together with Nick Witte I've applied the infinite log-gas formalism to predict the asymptotic expansion of $E_\beta^{\mathrm{hard}}(n;(0,s);a)$ and $E_\beta^{rmsoft}(n;(s,\infty))$.
- I've obtained rigorous large deviation formulas for a Laguerre ensemble finitization of $E_\beta^{\mathrm{hard}}(n;(0,s);a)$, and shown that the double scaling limit agrees with the infinite log-gas prediction.
- The latter makes essential use the Barnes double gamma function and satisfies the asymptotic functional equation

$$E_\beta^{\mathrm{hard}}(n;(0,s/\tilde{s}_\beta);\beta a/2) \underset{s\to\infty}{\sim} E_{4/\beta}^{\mathrm{hard}}(\beta(n+1)/2-1;(0,s/\tilde{s}_{4/\beta});a-2+4/\beta),$$

where $\tilde{s}_{4/\beta}(\beta/2)^2 = \tilde{s}_\beta$.

A conjecture

Define the Stirling modular form $\rho_2(1,\tau)$, which according to Shantani can be written

$$\rho_2(1,\tau) = (2\pi)^{3/4}\tau^{-1/4+(\tau+1/\tau)/12}e^{P(\tau)}\prod_{n=1}^{\infty}\frac{e^{Q(n\tau)}}{\Gamma(1+n\tau)},$$

with

$$P(\tau) = -\frac{\gamma}{12\tau}-\frac{\tau}{12}+\tau\zeta'(-1), \quad Q(x) = \left(\frac{1}{2}+x\right)\log x - x + \log\sqrt{2\pi}+\frac{1}{12x}.$$

We have

$$\log\tau^{\mathrm{bulk}}_{\beta/2} = \left(3-\frac{4}{3}(\beta/2+2/\beta)\right)\log 2 + 3\left(\frac{1}{2}\log 2\pi - \log\rho_2(1,2/\beta)\right)$$

and consequently

$$E^{\mathrm{bulk}}_{\beta}(0;(0,s/\pi)) \underset{s\to\infty}{\sim} \left(\frac{2}{\beta}\right)^{3/2}\widetilde{E^{\mathrm{bulk}}_{4/\beta}}(0;(0,\frac{\beta}{2}s/\pi)),$$

where $\widetilde{E^{\mathrm{bulk}}_{\beta}} \sim E^{\mathrm{bulk}}_{\beta}|_{s\mapsto -s}$.

A numerical realisation

Recall the exact result $E_2^{\mathrm{bulk}}(0;s) = \det(1 - K_{(0,s)})$, where K is the integral operator on $(0,s)$ with kernel $\frac{\sin \pi(x-y)}{\pi(x-y)}$.

Using methods advocated recently by Bornemann, together with the tanh-sinh quadrature rule, we can tabulate $r(s) = \frac{E_2^{\mathrm{b,as}}(0;(0,s))}{E_2^{\mathrm{bulk}}(0;(0,s))}$

s	$r(s)$
1	1.0046735914726577
2	0.9998383226940526
3	0.9999753765440204
4	0.9999961026171116
5	0.9999991096965057
6	0.9999997235559452
7	0.9999998946139279
8	0.9999999537746553
9	0.9999999775313906
10	0.9999999881794448