A domain wall theory for the prioritising exclusion process

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A queueing model

Customers waiting to be served:

1 2 1 2 1 1

- Priority class high (1) or low (2)

A queueing model

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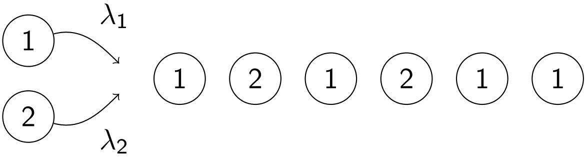

- Priority class high (1) or low (2)
- Arrive at rate λ_i customers / minute

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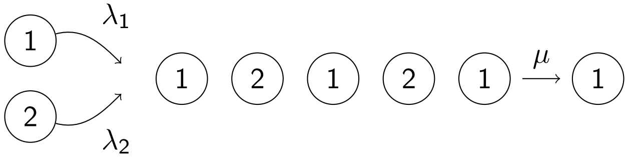

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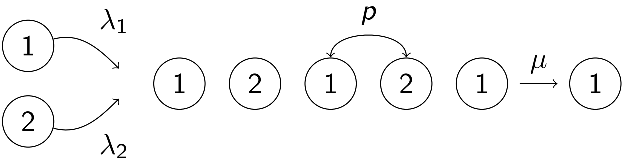
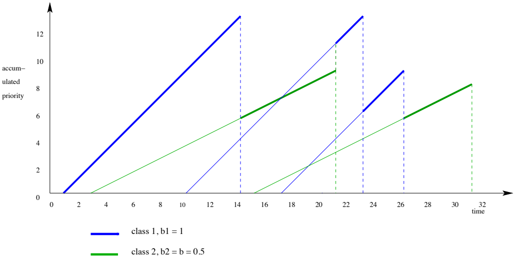


- Priority class high (1) or low (2)
- Arrive at rate λ_i customers / minute
- Served at rate μ customers / minute
- High priority overtake low at rate p places / minute

Motivation

A related model: the *accumulating priority queue*. Customers ordered according to priority

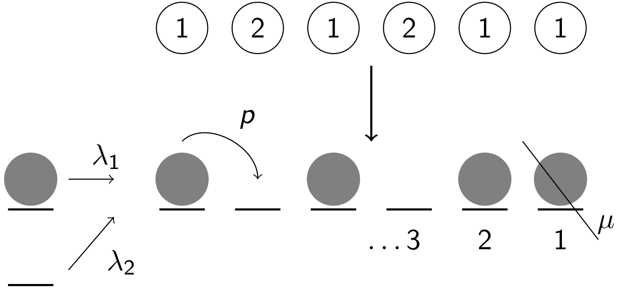
$$V = b_i(t - t_{\text{arrive}}), \quad i = 1, 2, \quad b_1 \geq b_2$$



Mean waiting time (Kleinrock, 1964) and complete waiting time distribution (Stanford, Taylor, Ziedins, 2011) are known.

The Prioritising Exclusion Process

The queue is equivalent to an exclusion process – the PEP



- Low priority $\rightarrow$ empty lattice site,
- High priority $\rightarrow$ filled lattice site
- Overtaking $\rightarrow$ particle hopping
- **But**, lattice length not fixed
- Specify a configuration by

$$\boldsymbol{\tau} = (\tau_n, \tau_{n-1}, \dots, \tau_1)_n, \qquad \tau_i \in \{0, 1\}$$

Bounded and unbounded queues

Let $\lambda = \lambda_1 + \lambda_2$.

- $\lambda > \mu$

$$\langle n \rangle \sim (\lambda - \mu) t \qquad \text{expected length is } \textit{unbounded}$$

- $\lambda < \mu$

$$\langle n \rangle = \frac{\lambda}{\mu - \lambda} \qquad \text{expected length is } \textit{bounded}$$

$$P_n = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n$$

P_n is the stationary length distribution, solution to

$$0 = \frac{d}{dt} P_0 = \mu P_1 - \lambda P_0$$

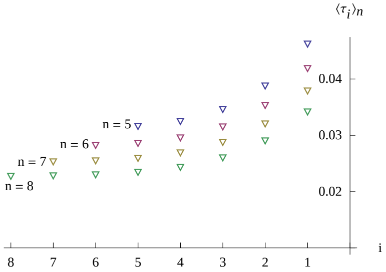
$$0 = \frac{d}{dt} P_n = \lambda P_{n-1} + \mu P_{n+1} - (\lambda + \mu) P_n, \quad n > 0$$

Density profiles

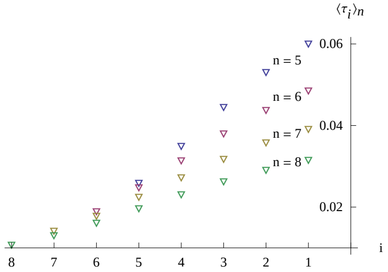
- Density by position (i) and queue length (n)

$$\langle \tau_i \rangle_n = P(\text{HI in place } i \text{ and queue length is } n)$$

- Simulation results show two distinct phases



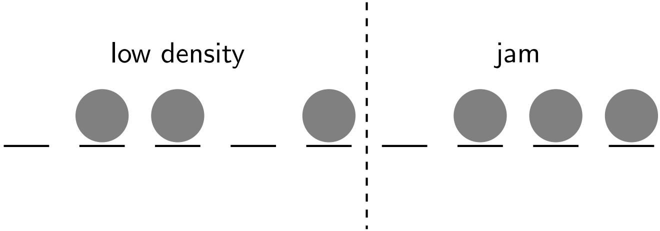
$\mu = 1, p = 1, \lambda_1 = 0.7, \lambda_2 = 0.2$



$\mu = 1, p = 3, \lambda_1 = 0.7, \lambda_2 = 0.1$

Domain wall model for the queue

The domain wall model is an idea borrowed from the ASEP



- Domain wall between *jam* and low density region located at first empty site from service end

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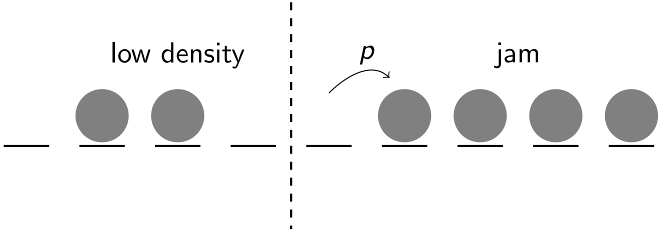

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- Jam grows when particle jumps on to the end

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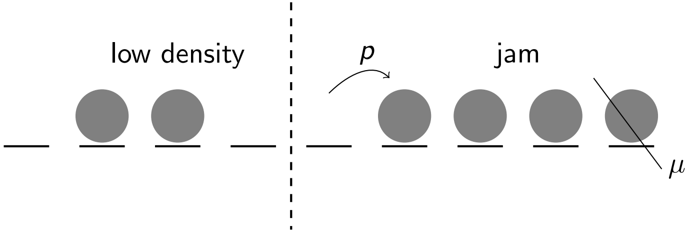
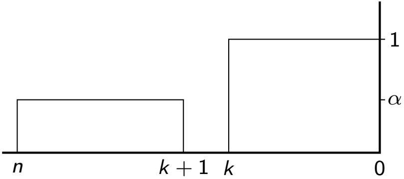


- Domain wall between *jam* and low density region located at first empty site from service end
- Jam grows when particle jumps on to the end
- Jam shrinks when service occurs

Domain wall model for the queue

Assume the region beyond the jam has constant low density α



$$P(\tau_i = 1|n,k) = \begin{cases} 1, & 1 \leq i \leq k \\ 0, & i = k+1 \\ \alpha, & k+2 \leq i \leq n \end{cases}$$

$P(n,k)$, probability of queue length n, jam length k, should satisfy

$$\frac{d}{dt}P(n,k) = \lambda P(n-1,k) + p\alpha P(n,k-1) + \mu P(n+1,k+1) \\ -(\lambda+\mu+p\alpha)P(n,k)$$

For $n > k+1$, $n,k > 0$.

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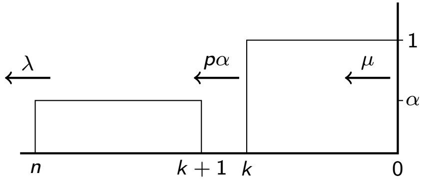

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For $n > k+1$, $n, k > 0$.

Simple domain wall model - boundary conditions

- Simple domain wall model is complicated by the boundary conditions – what happens when the jam reaches the arrival end?
- **Simplification**: consider the $n \to \infty$ limit

$$\frac{d}{dt}P(k) = p\alpha P(k-1) + \mu P(k+1) - (\mu + p\alpha)P(k)$$

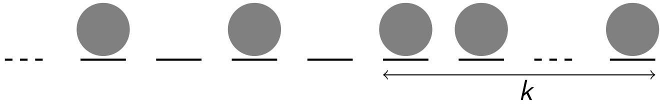
- The only boundary case is $k = 0$

The unbounded queue – an exact solution

Stationary probability of finite segment from an infinite queue:

$$P(\tau_m, \tau_{m-1}, \ldots, \tau_1) = \sum_{\tau_\infty, \ldots \tau_{m+1} = 0,1} P(\ldots, \tau_{m+1}, \tau_m, \tau_{m-1}, \ldots, \tau_1)$$

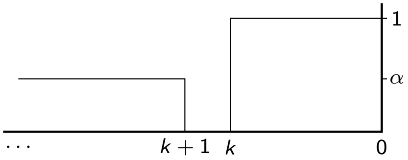
Example with length k jam



Exact equation:

$$\begin{aligned}
0 &= \frac{d}{dt} P(1,0,1,0,1^k) \\
&= pP(\mathbf{1},\mathbf{0},0,1,0,1^k) + pP(1,\mathbf{1},\mathbf{0},0,1^k) + pP(1,0,1,\mathbf{1},\mathbf{0},\mathbf{1^{k-1}}) \\
&\quad + \mu P(1,0,1,0,\mathbf{1^{k+1}}) + \mu P(1,0,1,0,1^k,\mathbf{0}) \\
&\quad - (\mu + 2p) P(1,0,1,0,1^k)
\end{aligned}$$

Domain wall ansatz



$$P(\tau_i = 1|k) = \begin{cases} 1, & 1 \le i \le k \\ 0, & i = k+1 \\ \alpha, & k+2 \le i \end{cases}$$

$$P(\tau_m, \tau_{m-1}, \ldots, \tau_{k+2}, 0, 1^k) = \alpha^{m_1}(1-\alpha)^{m-k-m_1-1} P_{\mathrm{jam}}(k)$$

$$m_1 = \sum_{i=k+2}^{m} \tau_i$$

Applied to the example

$$\begin{aligned} 0 &= \frac{d}{dt} P_{\mathrm{jam}}(k) \\ &= p\alpha P_{\mathrm{jam}}(k-1) + \mu P_{\mathrm{jam}}(k+1) + \mu(1-\alpha)\alpha^k P_{\mathrm{jam}}(0) \\ &\quad - (\mu + p\alpha) P_{\mathrm{jam}}(k) \end{aligned}$$

Almost identical to the simple domain wall dynamics

Recurrence for $P_{\text{jam}}(k)$

Combining with $k = 0$ case gives the recurrence

$$P_{\text{jam}}(k) = \frac{p\alpha}{\mu} P_{\text{jam}}(k-1) + \alpha^k P_{\text{jam}}(0), \qquad k \geq 1$$

Solution

$$\begin{aligned} P_{\text{jam}}(k) &= \sum_{i=0}^{k} \left(\frac{p\alpha}{\mu}\right)^{k-i} \alpha^i P_{\text{jam}}(0) \\ &= (p\alpha)^k \frac{\left(\frac{1}{p}\right)^{k+1} - \left(\frac{1}{\mu}\right)^{k+1}}{\frac{1}{p} - \frac{1}{\mu}} P_{\text{jam}}(0) \end{aligned}$$

$P_{\text{jam}}(0)$ determined by the normalisation condition

$$\sum_{k=0}^{\infty} P_{\text{jam}}(k) = 1$$

Constraint $p\alpha < \mu$: domain wall stays near service end

The arrival frame

Reference frame at the *arrival end* of the queue

For a finite segment at the arrival end, rate equation includes arrival terms

$$
\begin{aligned}
0 &= \frac{d}{dt} P(\tau_1, \tau_2, \ldots, \tau_m) \\
&= \ldots \tau_1 \lambda_1 P(\tau_2, \ldots, \tau_m) + (1 - \tau_1)\lambda_2 P(\tau_2, \ldots, \tau_m) \\
&\quad - \lambda P(\tau_1, \ldots, \tau_m) + \ldots
\end{aligned}
$$

The arrival frame solution

Assuming the jam stays far from the arrival end, the *arrival frame* ansatz is

$$P(\tau_1, \ldots, \tau_m) = \alpha^{m_1}(1-\alpha)^{m-m_1}$$

All cases reduce to

$$p\alpha^2 - (p+\lambda)\alpha + \lambda_1 = 0$$
$$\Rightarrow \alpha = \frac{p+\lambda-\sqrt{(p-\lambda)^2+4p\lambda_2}}{2p}$$

Note $p\alpha < \lambda$: domain wall stays far from arrival end

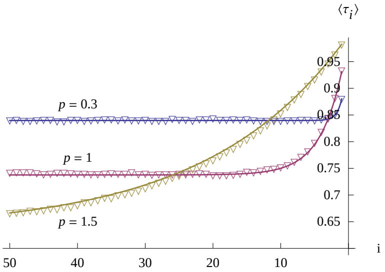
Exact solution vs Monte Carlo simulation

- Service frame density from domain wall solution

$$\langle \tau_i \rangle = \alpha \sum_{k-0}^{i-2} P_{\mathrm{jam}}(k) + 0 \times P_{\mathrm{jam}}(i-1) + 1 \times \sum_{k=i}^{\infty} P_{\mathrm{jam}}(k)$$

$$= \alpha + (1-\alpha)\left(\frac{p\alpha}{\mu}\right)^i$$

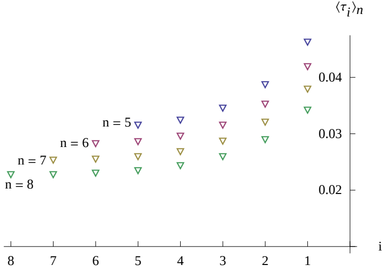
- Monte Carlo simulation, $\lambda_1 = 1.3$, $\lambda_2 = 0.2$, $\mu = 1$, $p = 0.3, 1, 1.5 \rightarrow p\alpha \simeq .25, .74, .95$



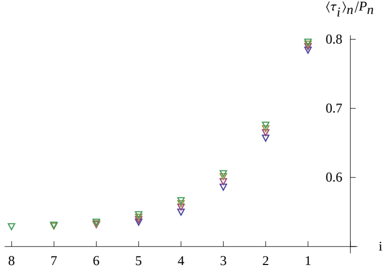
The bounded queue

Density profiles have strong dependence on the length distribution

- $\langle \tau_i \rangle_n \leq P_n, \qquad P_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$
- Can we factor out length dependence?



$\langle \tau_i \rangle_n$



$\langle \tau_i \rangle_n / P_n$

Domain wall ansatz with *length assumption*

$$P(\tau_n, \tau_{n-1}, \ldots, \tau_{k+2}, 0, 1^k; n) = P_n \alpha^{n_1} (1-\alpha)^{n-k-n_1-1} P^*_{\mathrm{jam}}(k)$$

Length assumption solution

Solution for domain wall away from boundaries

$$
\begin{aligned}
P^*_{\mathrm{jam}}(k) \qquad &= \sum_{i=0}^{k} \left(\frac{p\alpha}{\lambda}\right)^{k-i} \alpha^i P^*_{\mathrm{jam}}(0) \\
&= (p\alpha)^k \frac{\left(\frac{1}{p}\right)^{k+1} - \left(\frac{1}{\lambda}\right)^{k+1}}{\frac{1}{p} - \frac{1}{\lambda}} P^*_{\mathrm{jam}}(0),
\end{aligned}
$$

for $p\alpha < \lambda$, and

$$
\alpha = \frac{p + \mu - \sqrt{(p+\mu)^2 - 4p\frac{\lambda_1 \mu}{\lambda}}}{2p}.
$$

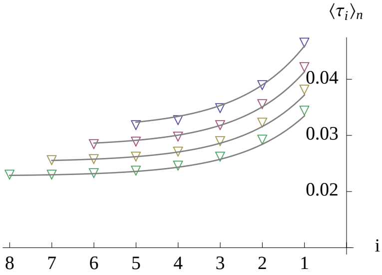
Solution fails if the jam reaches the arrival end, e.g.

$$
\frac{d}{dt} P(0, 1^{n-1}; n) \neq 0
$$

An approximate solution to an approximation of the problem $\rightarrow$ compare to simulations

Approximate solution vs Monte Carlo simulation

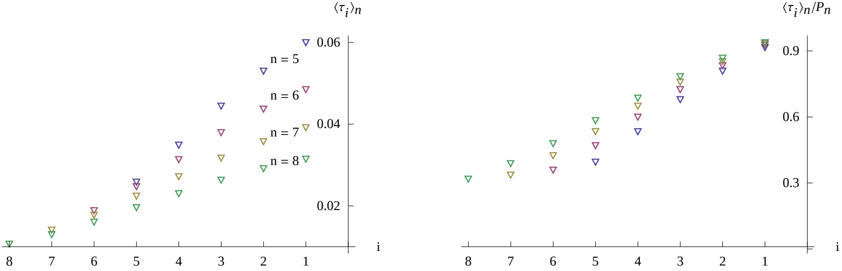
With $\mu = 1, p = 1, \lambda_1 = 0.7, \lambda_2 = 0.2 \Rightarrow p\alpha = 0.53$



When $p\alpha < \lambda$, domain wall solution with length assumption (solid lines) gives a very good approximation.

Monte Carlo results for $p\alpha > \lambda$

With $\mu = 1, p = 3, \lambda_1 = 0.7, \lambda_2 = 0.1 \Rightarrow p\alpha = 0.83$

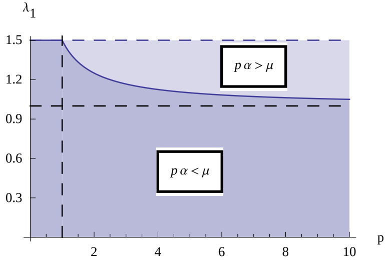


For $p\alpha > \lambda$, the domain wall solution does not apply and even the length assumption is clearly invalid.

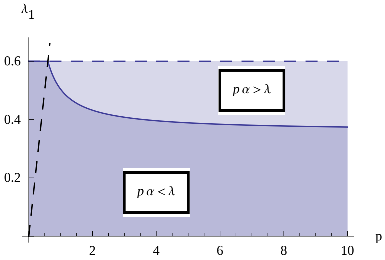
Stationary phase diagram

- Fixed λ, and $\mu = 1$
- Phase diagram for $0 \leq \lambda_1 \leq \lambda$, $p > 0$

Unbounded queue ($\lambda = 1.5$)



Bounded queue ($\lambda = 0.6$)



$p\alpha < \mu \Rightarrow$

$$\lambda_1 < \min\{1 + \frac{\lambda - 1}{p}, \lambda\}$$

$p\alpha < \lambda \Rightarrow$

$$\lambda_1 < \min\{\lambda^2 \left(1 + \frac{1 - \lambda}{p}\right), \lambda\}$$

Conclusion

From a simple physical picture, the domain wall idea, we

- Found the exact solution for the unbounded queue;
- Found a good approximation for the bounded queue;
- Characterised the stationary behaviour, in both cases, by the rate at which the jam of high priority customers grows.

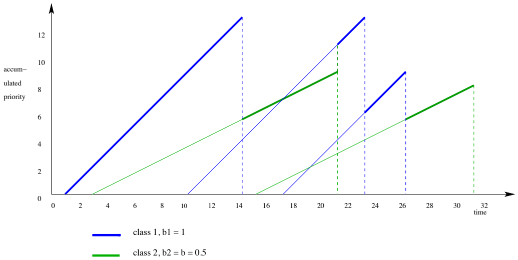
Future work:

- Compute waiting times for the PEP and compare to the accumulating priority queue (APQ).
- Is the change in stationary behaviour we see in the PEP reflected in the APQ?
- Can we find an exact solution for the bounded PEP?

A related queueing model

The PEP is related to the *accumulating priority queue*: customers ordered according to priority

$$V = b_i(t - t_{\text{arrive}})$$



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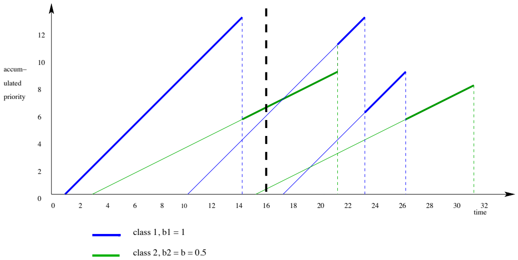

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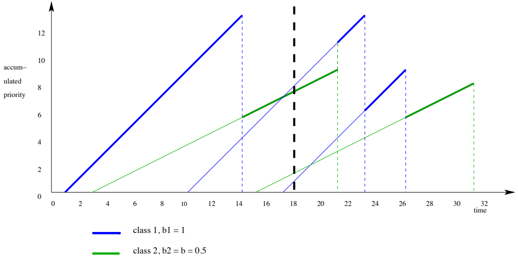

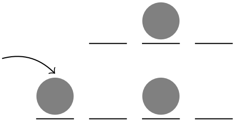

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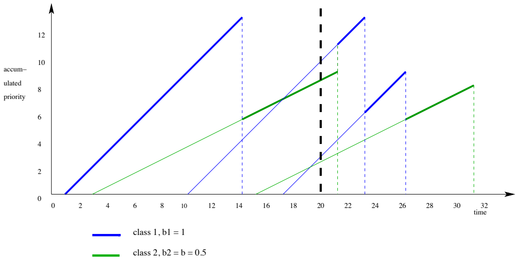

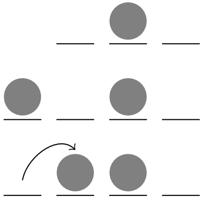

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