Endings and Beginnings: The Story of Non-Intersecting Paths

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Outline

Motivation

- Discrete models for real world phenomena:
  Lattice Paths are a simple model for polymers

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  Lattice Paths are a simple model for polymers
- Some results are of interest directly
- The methods lead to producing efficient algorithms, of much use in computing/data mining

Combinatorial Methods

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Find relationships between different sets of objects

Deduce formulae/prove known results.

Bijections

Definition

A bijection is a function $\Gamma \colon A \to B$ such that Γ is:

- well-defined;
- injective, $\Gamma(a) = \Gamma(a') \implies a = a'$; and
- surjective, $\forall b \in B,\ \exists a \in A \colon b = \Gamma(a)$.

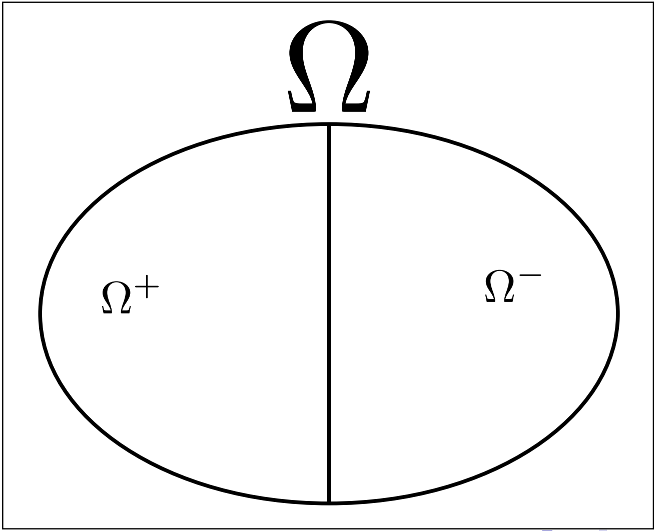
Involutions

Definition

Consider a signed set $\Omega = \Omega^+ \cup \Omega^-$ where $\Omega^+ \cap \Omega^- = \emptyset$.
$\varphi : \Omega \to \Omega$ is an involution if

1. $\varphi^2 = 1$; and
2. for all $a \in \Omega$, φ is either fixed or sign-reversing.

Involutions



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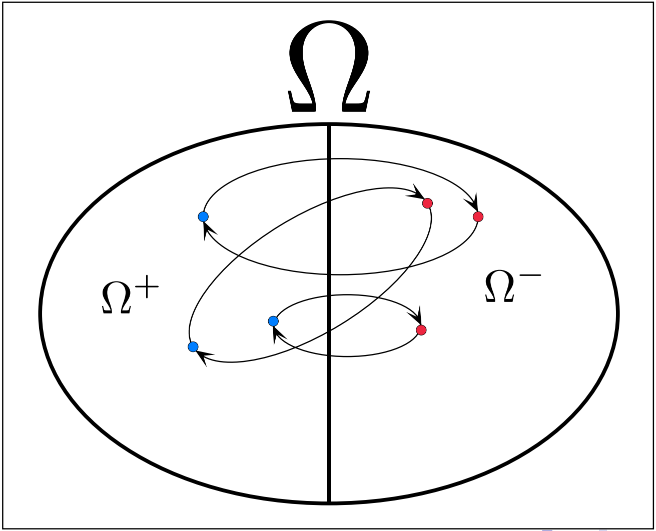

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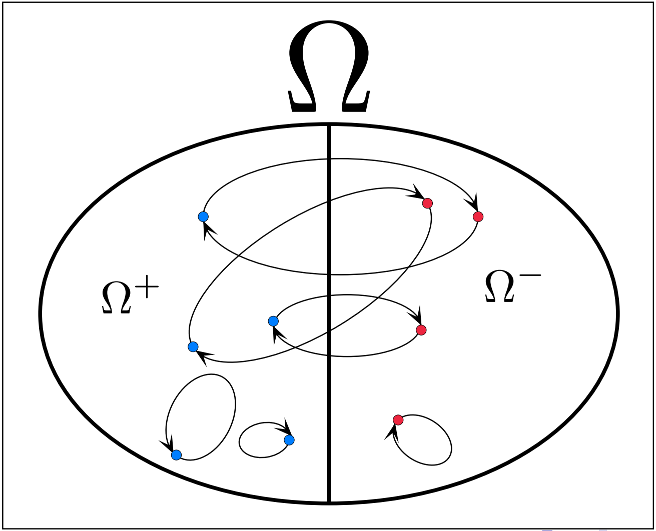

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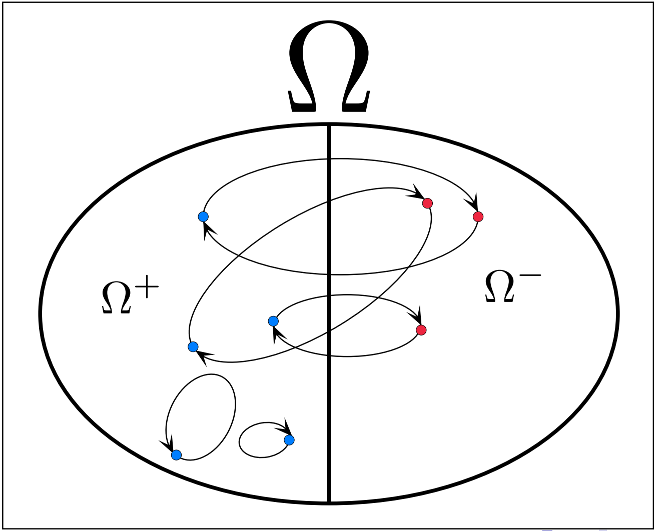

Lattice Paths

Definition

A path p on the integer lattice is a sequence of vertices $p = v_0 v_1 \dots v_t$ such that $v_i \in \mathbb{Z} \times \mathbb{Z}$ and $(v_{i+1} - v_i) \in S$, we call S the *step set* of the paths.

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Definition

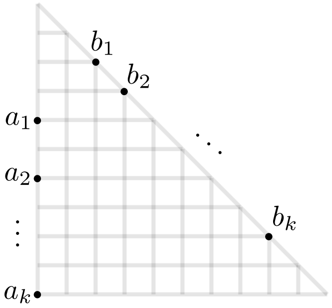
A binomial path is a path with the step set $S = \{(0,1),(1,0)\}$.

The number of binomial paths of length n with k horizontal steps is given by the binomial coefficient

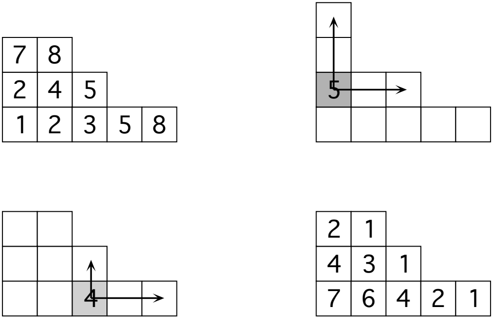
$$\binom{n}{k}$$

Sets of Lattice Paths

Interested in sets of lattice paths with the geometry:



Young Tableaux



Young tableaux have

- shape μ
- content c_x for each cell x
- hook lengths h_x for each cell x

Theorem on Non-Intersecting Paths

Theorem (Gessel-Viennot, Lindström)

Consider a directed acyclic graph $G = (V, E)$*, and let* $|a_i \rightsquigarrow b_j|$ *be the number of directed paths from* a_i *to* b_j *where* $a_i, b_j \in V$*. If either every path* $a_i \rightsquigarrow b_i$ *intersects every path* $a_j \rightsquigarrow b_j$*; or every path* $a_i \rightsquigarrow b_j$ *intersects every path* $a_j \rightsquigarrow b_i$*, then*

$$|\mathcal{N}(\mathbf{a}|\mathbf{b})| = \det\left([|a_i \rightsquigarrow b_j|]_{i,j \in [n]}\right).$$

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Theorem (Gessel & Viennot)

Let $a_i = a + (i-1)$, $\mu = [p(\mathbf{b})]^\star$, $C_a(\mu) = \prod_{x\in\mu}(a + c_x)$ *and* $H(\mu)$ *be the product of the hook lengths of* μ. *Then:*

$$|\mathcal{N}(\mathbf{a}|\mathbf{b})| = \frac{C_a(\mu)}{H(\mu)}$$

The Hook Length Formula

- Famous result due to Gessel and Viennot, and independently due to Lindström
- Gessel-Viennot proof was almost entirely combinatorial
- One lemma was only able to be proved algebraically
- This "missing lemma" is also known as the 2nd Remmel Recurrence
- Later an implicit combinatorial proof was given using the Garsia-Milne Method

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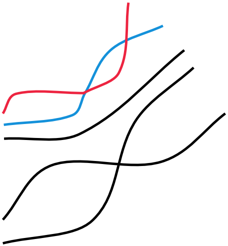
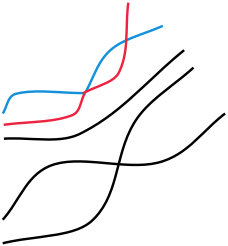


(1)(2)(3)(45)



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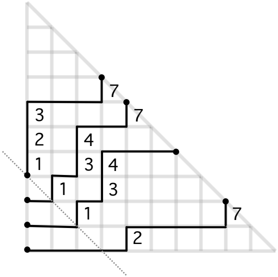

7	7		
3	4	4	
2	3	3	7
1	1	1	2

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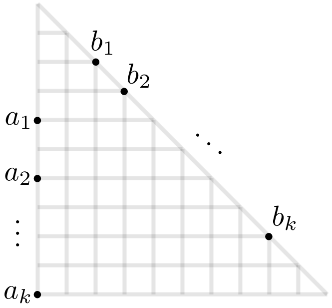
4. Hook Length Formula

Let the number of non-intersecting configurations of k binomial paths on the geometry below be:

$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k}$$



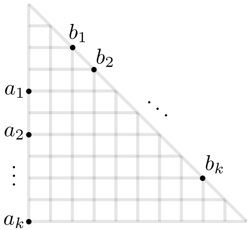
The Missing Lemma

Theorem (Fijn & Brak)

If $b_1 \neq 0$ *then*

$$b_1 b_2 \cdots b_k \binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = a_1 a_2 \cdots a_k \binom{a_1 - 1, \ldots, a_k - 1}{b_1 - 1, \ldots, b_k - 1}$$

Combinatorial Interpretation



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- Each path has b_i horizontal steps, and a_i total steps.
- LHS we mark one horizontal edge on each path.
- RHS we have one fewer horizontal edge on each path, and mark one vertex on each path.

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Asymptotic formulae

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Summation forms

$$\sum_{\mathbf{p}\in\mathcal{O}_t^\star} w(\mathbf{p}) = \sum_{\sigma\neq 1}\sum_{\substack{I^\star(\sigma)\\ k\geq 1}}\prod_{i=1}^{N}\omega^k(-1)^{|\mathcal{I}_\sigma|+k_<^+}\binom{t-k^\star}{b_{\sigma_i}-a_i-k-k_<^+ + k_>^+}$$

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Product forms

$$\binom{a_1,\ldots,a_k}{b_1,\ldots,b_k} = \prod_{i=1}^{k} \binom{a_i}{b_i}$$

Future Research

Previously no known combinatorial proofs for product forms.

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Vast array of product forms which may be soluble by these methods.

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Alternating Sign Matrices (and various symmetry classes thereof).