Topological Order in Spin Systems

Paul Fendley

Much ado about topological order

Systems with topological order in 2+1 dimensions typically have anyonic/fractionalized/spin-charge separated excitations.

These quasiparticles can even have non-abelian statistics, i.e. when ``braided'' around each other, the system can change state.

Local perturbations don't affect statistics, so this gives promise for topologically protected quantum computing.

What is topological order?

- Conceptually useful definition: the number of ground states depends on topological properties (e.g. genus) of space.

- Common (although not required) characteristic:

gapless edge modes

Free fermion cases are now well understood

For example, topological insulators and superconductors made from free fermions in arbitrary dimensions have been classified

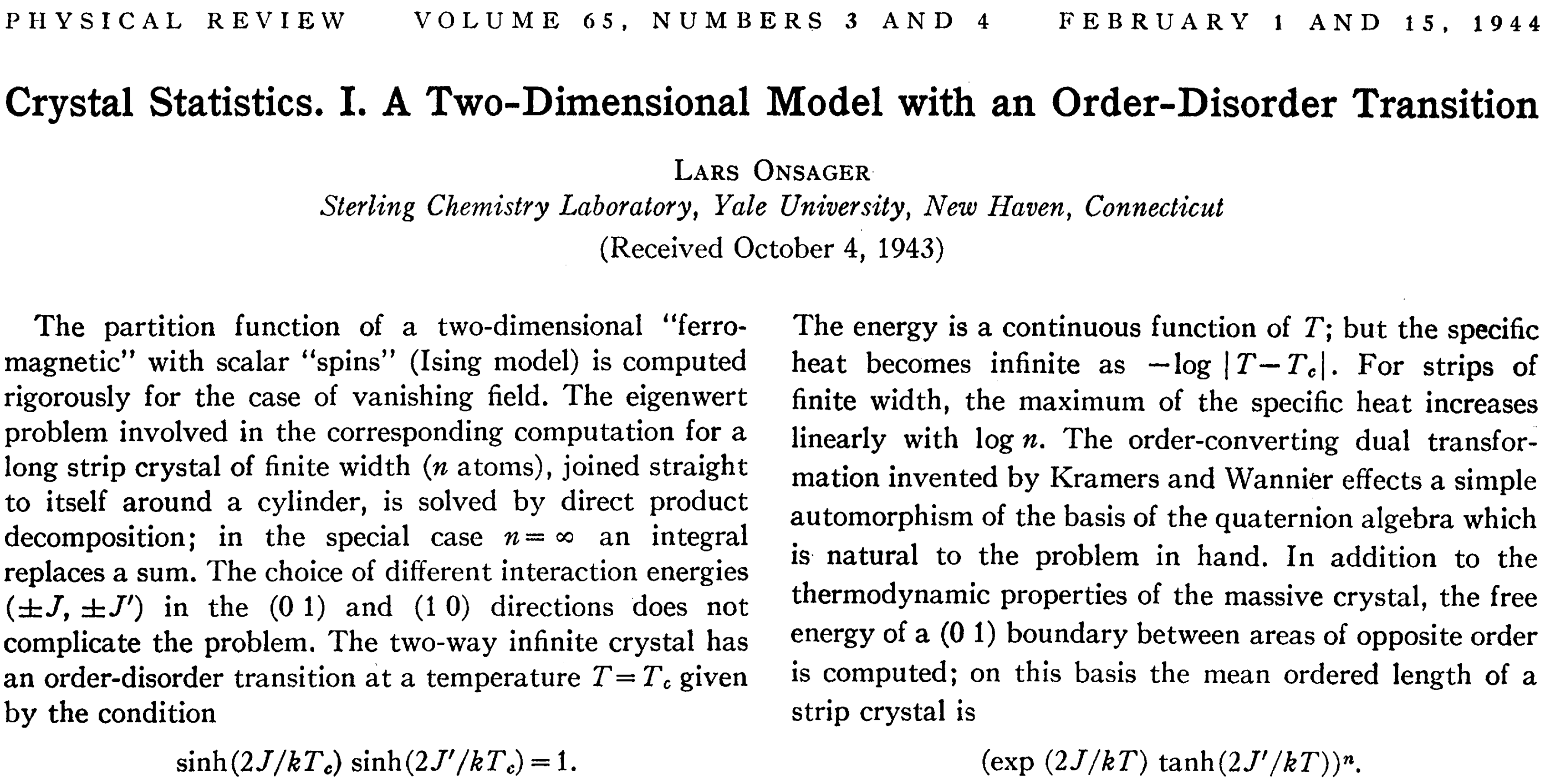
via K theory: Kitaev; via edge theories: Ryu, Schnyder, Furusaki, Ludwig

A well-known example is the integer quantum Hall effect.

This, however, is not the simplest example...

The quantum Ising chain

Yes, the one Onsager solved in the '40s …



PHYSICAL REVIEW VOLUME 65, NUMBERS 3 AND 4 FEBRUARY 1 AND 15, 1944

Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

LARS ONSAGER
Sterling Chemistry Laboratory, Yale University, New Haven, Connecticut

(Received October 4, 1943)

The partition function of a two-dimensional "ferromagnetic" with scalar "spins" (Ising model) is computed rigorously for the case of vanishing field. The eigenwert problem involved in the corresponding computation for a long strip crystal of finite width (n atoms), joined straight to itself around a cylinder, is solved by direct product decomposition; in the special case $n=\infty$ an integral replaces a sum. The choice of different interaction energies $(\pm J, \pm J')$ in the (0 1) and (1 0) directions does not complicate the problem. The two-way infinite crystal has an order-disorder transition at a temperature $T=T_c$ given by the condition

$$\sinh(2J/kT_c)\sinh(2J'/kT_c)=1.$$

The energy is a continuous function of T; but the specific heat becomes infinite as $-\log|T-T_c|$. For strips of finite width, the maximum of the specific heat increases linearly with $\log n$. The order-converting dual transformation invented by Kramers and Wannier effects a simple automorphism of the basis of the quaternion algebra which is natural to the problem in hand. In addition to the thermodynamic properties of the massive crystal, the free energy of a (0 1) boundary between areas of opposite order is computed; on this basis the mean ordered length of a strip crystal is

$$(\exp(2J/kT)\tanh(2J'/kT))^n.$$

57 years later, Kitaev made a trivial-but-profound observation:

Fermions exist in nature

The easiest way to solve the 2d Ising model is to follow Kaufmann and map it on to free fermions. In the 1d quantum chain limit, this amounts to a Jordan-Wigner transformation.

If your physical system is comprised of spins, then this is a mathematical trick.

But if your physical system is comprised of fermions…

The quantum Ising chain

The Kitaev chain

Unpaired Majorana fermions in quantum wires

Alexei Yu. Kitaev*
Microsoft Research
Microsoft, #113/2032, One Microsoft Way,
Redmond, WA 98052, U.S.A.
kitaev@microsoft.com

27 October 2000

Abstract

Certain one-dimensional Fermi systems have an energy gap in the bulk spectrum while boundary states are described by one Majorana operator per boundary point. A finite system of length L possesses two ground states with an energy difference proportional to $\exp(-L/l_0)$ and different fermionic parities. Such systems can be used as qubits since they are intrinsically immune to decoherence. The property of a system to have boundary Majorana fermions is expressed as a condition on the bulk electron spectrum. The condition is satisfied in the presence of an arbitrary small energy gap induced by proximity of a 3-dimensional p-wave superconductor, provided that the normal spectrum has an odd number of Fermi points in each half of the Brillouin zone (each spin component counts separately).

In terms of fermions, the quantum Ising chain includes a Cooper-pairing interaction $c_i^\dagger c_{i+1}^\dagger + c_i c_{i+1}$, so fermion number is only conserved mod 2.

As a consequence, the fermions are Majorana: they have a single fermi point. In old-folks language, there is no fermion doubling.

Open boundary conditions in the Ising ordered phase yield edge zero modes in the fermion picture.

Ising order corresponds to topological order!

And this has probably been seen experimentally!

Majoranas are not the end of the story

Interacting systems such as the fractional quantum Hall effect exhibit still more interesting behavior: charge fractionalization, universal topological quantum computation…

Understanding the physics here is much more difficult.

A basic thing to do is to add local fermion interactions. In 1d, this results in a classification very similar to that of free-fermi systems.

Fidkowski and Kitaev

A different approach to topological order for interacting systems

- 1d ``spin'' systems with $\mathbb{Z}_n$ symmetry can be mapped onto parafermions. These in general are not perturbations of free fermions, and are strongly interacting.

- Nonetheless, in 1d they can have edge zero modes, just like Majorana/Ising.

- A byproduct is that we learn something very interesting about the integrable chiral Potts model.

- Moreover, there are 2d spin models, a la Kitaev honeycomb model, that are parafermions plus a background gauge field.

Outline

- Edge/zero modes in the Ising/Majorana chain

- Edge/zero modes in the 3-state (chiral) clock chain using parafermions

  only if spatial-parity and time-reversal symmetries are broken!

- Parafermions, the chiral Potts model and the Onsager algebra

  coming full circle!

- 2d quantum systems with $\mathbb{Z}_n$ symmetry

  topological order here?

Quantum chains

- Both for physics and technical reasons, it is better here to study one-dimensional quantum chains instead of 2d classical models

- The 2d classical Ising model consists of a $\mathbb{Z}_2$ "spin" variable at each site of some 2d lattice, with nearest-neighbor interactions.

- The quantum Ising Hamiltonian for an *L*-site chain then acts on a 2^L-dimensional vector space. This comes from taking an extremely anisotropic limit of the classical transfer matrix.

How to fermionize the quantum Ising chain

$$H = -\sum_j \left[\underset{\underset{\text{flip term}}{\uparrow}}{f\sigma_j^x} + J\underset{\underset{\text{interaction}}{\uparrow}}{\sigma_j^z\sigma_{j+1}^z} \right]$$

Critical point is when $J = f$, ordered phase is $J > f$.

$\mathbb{Z}_2$ symmetry operator is flipping all spins:

$$\prod_j \sigma_j^x$$

Jordan-Wigner transformation in terms of Majorana fermions

$$\psi_j = \sigma_j^z \prod_{i<j} \sigma_i^x \, , \qquad \chi_j = \sigma_j^y \prod_{i<j} \sigma_i^x$$

string

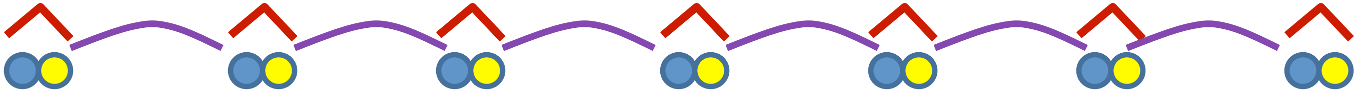
$$\{\psi_i , \psi_j\} = \{\chi_i , \chi_j\} = 2\delta_{ij} , \qquad \{\psi_i , \chi_j\} = 0$$

$\mathbb{Z}_2$ symmetry measures even or odd number of fermions:

$$(-1)^F = \prod_j \sigma_j^x = \prod_j (i\psi_j \chi_j)$$

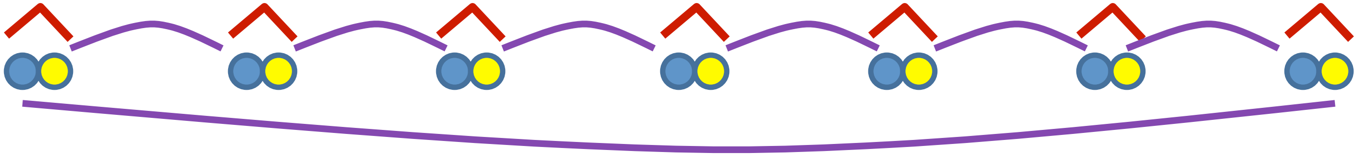
The Hamiltonian in terms of fermions

- with free boundary conditions:



$$H = if \sum_{j=1}^{N} \psi_j \chi_j + iJ \sum_{j=1}^{N-1} \chi_j \psi_{j+1}$$

- with periodic boundary conditions on the fermions:



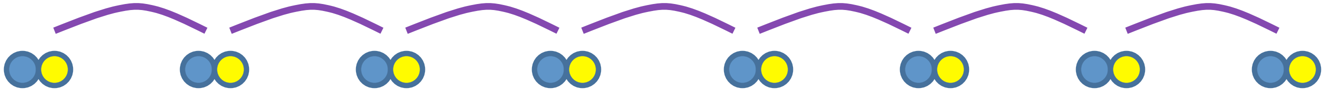
$$H = i \sum_{j=1}^{N} [f \psi_j \chi_j + J \chi_j \psi_{j+1}]$$

A catch: when written in terms of spins, this is twisted by $-(-1)^F$

Extreme limits:

- $J = 0$ (disordered in spin language):

- $f = 0$ (ordered in spin language):



The fermions on the edges, χ_1 and ψ_N, do not appear in H when $f = 0$. They commute with H!

Exact edge zero modes in the ordered phase

- When $f = 0$, the operators χ_1 and ψ_N map one ground state to the other – they form an exact zero mode.

- The gapless edge modes persist for all $f < J$: the series

$$\chi_1 + \frac{f}{J}\chi_3 + \left(\frac{f}{J}\right)^2 \chi_5 + \ldots$$

commutes with H.

- In the ordered phase $f < J$, this is is localized near the edge.

Spin order → topological order

- For non-abelian topological order degeneracies are necessary, so that braiding can change states.

- The 1d edge zero mode commutes with H, but not with $(-1)^F$. It thus means that all states are two-fold degenerate.

- Only way to change between states is to add a fermion, or to do a non-local operation (act at both ends).

- The non-local transformation between spins and fermions is a feature! It makes the degeneracy robust.

How does one characterize 1d topological order with periodic boundary conditions?

Simple way for 1d: can show it depends on $(-1)^F$ of ground state.

Even fancier way: compute sign of Pfaffian.

For the experts: this is the 1d analog of the 2d Chern number.

On to the $\mathbb{Z}_n$ case:

- Fradkin and Kadanoff showed long ago that 2d spin models with $\mathbb{Z}_n$ symmetry can be written in terms of parafermions.
- Fateev and Zamolodchikov found integrable critical self-dual lattice spin models with $\mathbb{Z}_n$ symmetry. Later they found an elegant CFT description of the continuum limit.
- Read and Rezayi constructed fractional quantum Hall wavefunctions using the CFT parafermion correlators.

The 3-state (chiral) clock model

The quantum chain version of the 3-state clock/Potts model:

$$H = -\sum_j \left[f(\tau_j + \tau_j^\dagger) + J(\sigma_j^\dagger \sigma_{j+1} + \mathrm{h.c.}) \right]$$

flip is now "shift"   "clock" potential

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\boxed{\tau\sigma = e^{2\pi i/3}\sigma\tau}$$

A quantum information tangent: ``SIC-POVM'' conjecture

- Consider *n*-dimensional complex unit vectors:

$$z = (z_1, z_2, \ldots z_n); \qquad z^\dagger \cdot z = 1$$

- The conjecture is that there are n^2 vectors, $z^{(1)}, \ \ldots \ , z^{(n^2)}$ that are ``equidistant'' -- their inner products are all the same:

$$| z^{(i)} \cdot z^{(j)} |^2 = \frac{1}{n+1}; \qquad i \neq j$$

- It seems that these are all given by acting by shift and clock on a single vector!

Zauner; Renes, Blume-Kohout, Scott and Caves

Define parafermions just like fermions:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain,

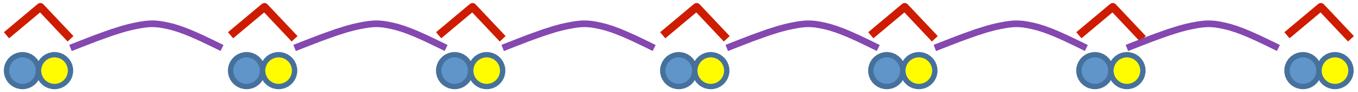
$$\psi_j = \sigma_j \prod_{i<j} \tau_i, \qquad \chi_j = \tau_j \sigma_j \prod_{i<j} \tau_i$$

$$\psi^3 = \chi^3 = 1, \quad \psi^2 = \psi^\dagger, \quad \chi^2 = \chi^\dagger$$

Instead of anticommutators, for $i < j$ and $\gamma = \chi$ or ψ :

$$\gamma_i \gamma_j = e^{2\pi i/3} \gamma_j \gamma_i$$

The Hamiltonian in terms of parafermions:

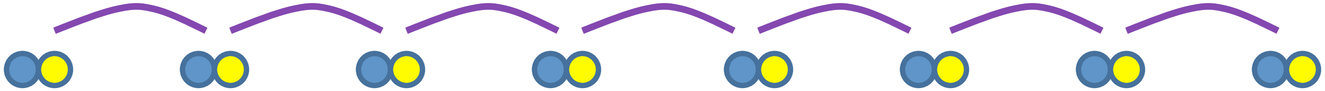


^ $= f(\psi_j^\dagger \chi_j + \chi_j^\dagger \psi_j)$ ⌒ $= J(\psi_{j+1}^\dagger \chi_j + \chi_j^\dagger \psi_{j+1})$

shift term potential

These parafermions are not perturbations of free fermions – they cube to 1. The model isn't even integrable unless $J = f$.

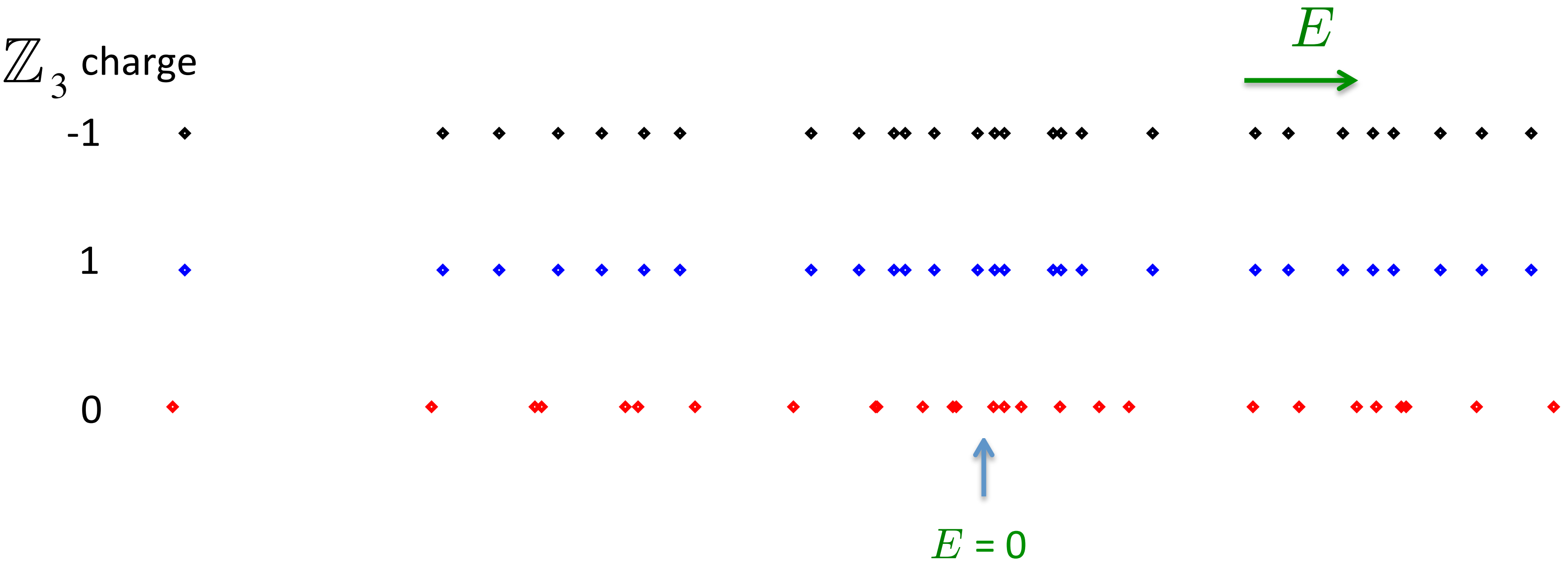
However, when f = 0, there are edge zero modes!



Does the zero mode remain for $J > f > 0$?

We can’t cheat like in the Ising chain and just solve the model.

Since χ_1, ψ_L involve the shift, a zero mode maps between $\mathbb{Z}_3$ sectors. For 4 sites and *f=J/2*, the spectrum is



Doesn’t look like there is a zero mode…

So let’s think about a maybe-easier problem: periodic boundary conditions

A parafermion ``shift’’ mode shifts the energy uniformly between $\mathbb{Z}_3$ sectors, i.e.

$$[H,\Psi]=(\Delta E)\,\Psi$$

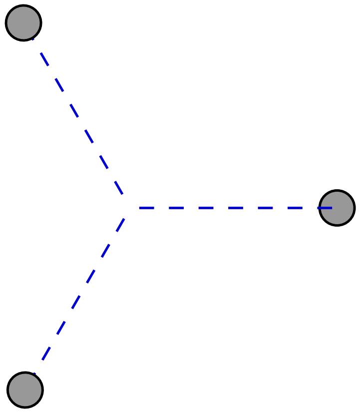
If $\Delta E=0$ then we have a zero mode.

There is a shift mode only if the couplings obey an interesting constraint!

Generalize to the chiral clock model:

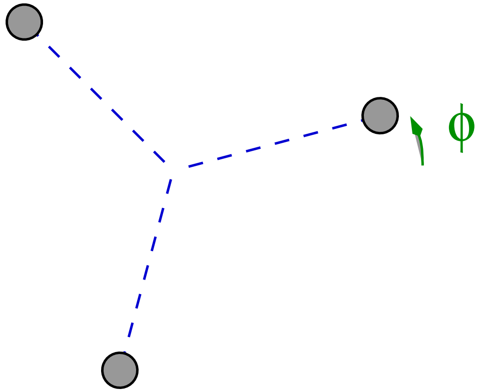
For classical spin variables $s_j = 1, e^{2\pi i/3}, e^{-2\pi i/3}$, the most general nearest-neighbor interaction is

$$-J(s_j^* s_k e^{i\phi} + s_j s_k^* e^{-i\phi})$$



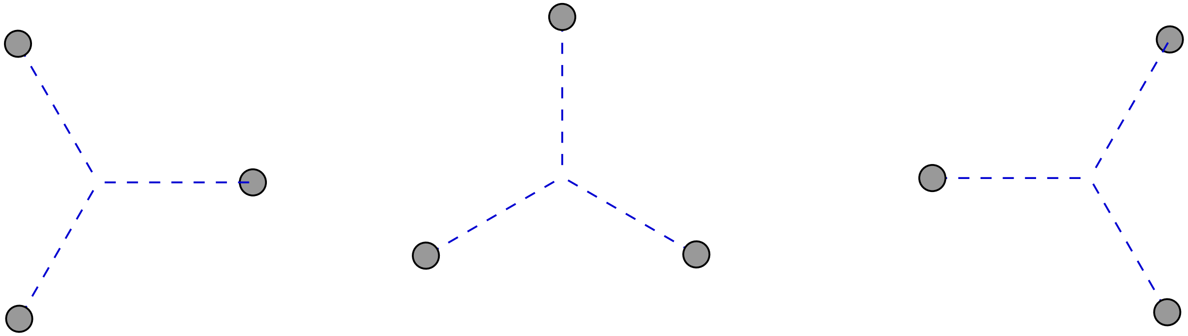
ferromagnet:

$$\phi = 0$$



Chiral interactions:

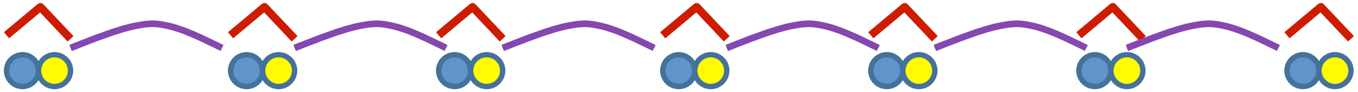
$$\phi \neq 0 \ \mathrm{mod} \pi / 3$$



ferromagnet — symmetric — antiferromagnet

$\phi = 0$ $\qquad$ $\phi = \pi / 6$ $\qquad$ $\phi = \pm\pi / 3$

In the quantum Hamiltonian:



^ $= f(e^{i\phi}\psi_j^\dagger \chi_j + e^{-i\phi}\chi_j^\dagger \psi_j)$

⌒ $= J(e^{i\theta}\psi_{j+1}^\dagger \chi_j + e^{-i\theta}\chi_j^\dagger \psi_{j+1})$

Look for a shift mode in the chiral model linear in the parafermions:

$$\Psi = \sum_j \left[\alpha_j \psi_j + \beta_j \chi_j\right]$$

Then a 10-minute computation finds an exact shift mode if the couplings obey:

$$f \cos(3\phi) = J \cos(3\theta)$$

$$f\cos(3\phi) = J\cos(3\theta)$$

This calculation is the world's easiest way of finding the couplings of the integrable chiral Potts chain.

Howes, Kadanoff and den Nijs; von Gehlen and Rittenberg; Albertini, McCoy, Perk and Tang; Baxter; Bazhanov and Stroganov

The integrable chiral Potts model is quite peculiar. The Boltzmann weights of the 2d classical analog are parameterized by higher genus Riemann surfaces instead of theta functions. They satisfy a generalized Yang-Baxter equation with no difference property. They are also 2d reductions of solvable 3d classical models.

The “superintegrable” line $\theta = \phi = \pi / 6$ is very special. Here the shift mode occurs for all values of f and J.

This is the symmetric case, halfway between ferro and antiferromagnet, where the spectrum is invariant under $H \rightarrow -H$.

Taking commutators of the these, one finds remarkable simplifications. They satisfy the Onsager algebra.

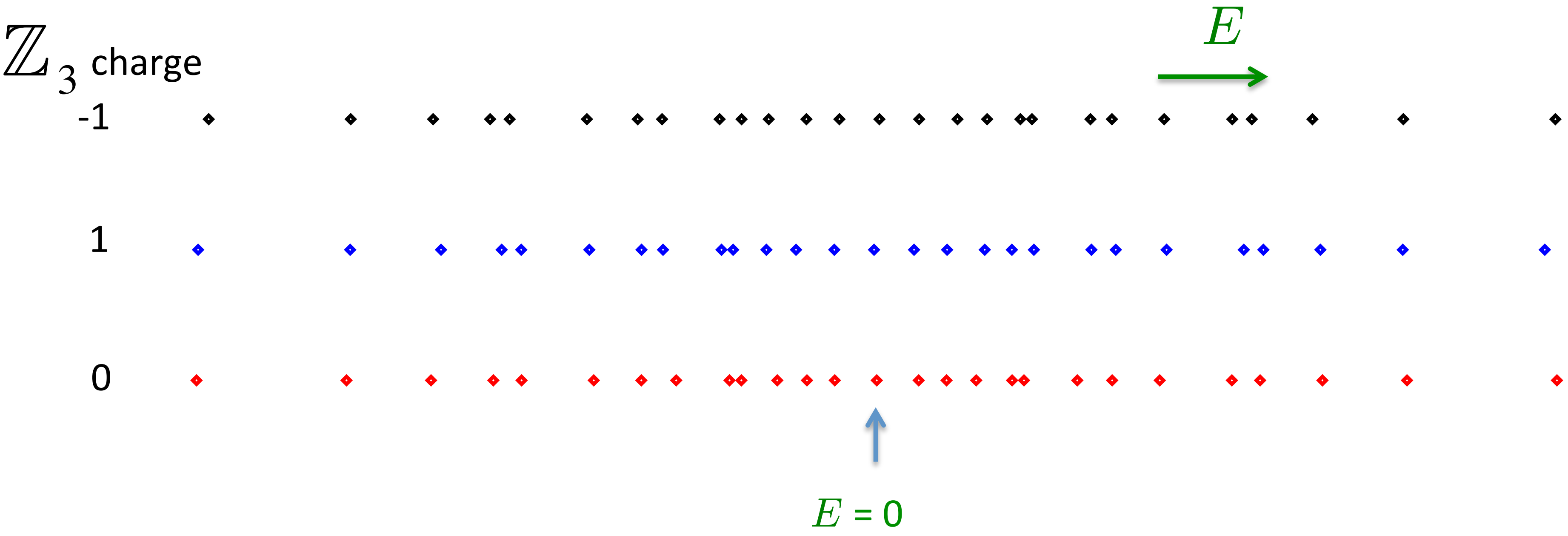
The identical algebra Onsager used to solve the Ising model originally also occurs in the chiral Potts models!

This allows the explicit construction of an infinite series of quantities commuting with the Hamiltonian.

von Gehlen and Rittenberg

Back to the edge

For 4 sites and *f=J/2*, the spectrum of open chain with $\phi = \theta = \pi / 6$:



The levels in three sectors get exponentially close as the number of sites is increased. Looks like the edge zero mode remains for *f/J*>0!

Finding the edge zero is not easy as with Ising. We can try to find it iteratively.

Split the Hamiltonian into flip terms and potential terms:

$$H = fT + JV$$

The parafermions on the edge obey

$$[V,\chi_1] = [V,\psi_L] = 0\ .$$

Then if there is an *X* such that

$$[T,\chi_1] = [V,X]$$

The zero mode on the left edge becomes

$$\chi_1 - \frac{f}{J} X + \ldots$$

$$[T, \chi_1] = [V, X]$$

Thus the question is: can *V* be ``inverted” to find *X* ?

For Ising, *T* and *V* are are free-fermion bilinears, so it’s easy:

$$\chi_1 + \frac{f}{J}\chi_3 + \left(\frac{f}{J}\right)^2 \chi_5 + \ldots$$

But parafermion commutation relations are not so nice:

$$[\chi_j^\dagger \psi_j, \psi_j] \propto \chi_j^\dagger \psi_j^\dagger$$

Say we want $[V, X] = \psi_1$. Then *X* must be a linear combination

$$X = A\psi_1 + B\chi_2 + C\psi_1^\dagger\chi_2^\dagger$$

so that $[V, X] = A'\psi_1 + B'\chi_2 + C'\psi_2^\dagger\chi_2^\dagger$ with

$$\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = 2i \begin{pmatrix} 0 & -e^{i\theta} & e^{-i\theta}\bar{\varpi} \\ e^{-i\theta} & 0 & -e^{i\theta}\bar{\varpi} \\ -e^{i\theta}\omega & e^{-i\theta}\omega & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$.

V can be inverted if this matrix can be inverted!

The determinant is $-16J^3\sin(3\theta)$, non-zero for the chiral case

$$\theta \neq 0 \ \mathrm{mod} \pi / 3$$

No edge zero mode for ferromagnet or antiferromagnet!

expansion parameter is $$\frac{f}{2J\sin(3\theta)}$$

Zero mode is $\chi_1 - \ 2if\, e^{-i\phi} X + \ 2if\, e^{i\phi} \chi_1^\dagger Y + \ldots ,$

$$X = \frac{1}{4J\sin(3\theta)}(\psi_1 + e^{2i\theta}\chi_2 + e^{-2i\theta}\omega\psi_1^\dagger\chi_2^\dagger)\,, \quad Y = -\frac{1}{4J\sin(3\theta)}(\psi_1^\dagger + e^{-2i\theta}\chi_2^\dagger + e^{2i\theta}\omega\psi_1^\dagger\chi_2^\dagger)$$

Perhaps not surprisingly, I do not have a closed-form expression for the edge zero mode in general.

Brute force not only gets extremely unwieldy, but after next order, no longer works!

But using by treating this matrix as a sort of Hamiltonian on the vector space of all parafermion operators, one can do degenerate perturbation theory and prove that the edge zero mode exists to all orders, as long as the interactions are chiral!

He didn’t know this orginally, but in the Ising case, the Onsager algebra is simply that of (zero-momentum) fermion bilinears!

Thus a miracle of the superintegrable chiral Potts model is that despite its not being a free-fermion theory, the algebra generated by the two parts of the Hamiltonian is identical to that of fermion bilinears.

In the $\mathbb{Z}_n$ case, maybe parafermions can be used to find a Pfaffian-ish formula to detect topological order ?!? The Read-Rezayian?!?

Topological order in two spatial dimensions

Even more remarkably, these correspondences can be extended to 2d quantum models, where anyons exist!

The Kitaev honeycomb model is a spin model with nearest-neighbor interactions. By exploiting a non-obvious gauge symmetry, it can be mapped on to free fermions with a background gauge field.

Breaking time-reversal symmetry results in topological order!

A similar model can be found for parafermions. However, it's not exactly solvable, so is it topologically ordered?

Correspondence to quantum loop models?

Questions

- The 1d case hints that in 2d we should include these chiral phases. But how to do this precisely?

- Is there a ``Read-Rezayian'' formula for the parafermions generalizing the Pfaffian/Chern number for fermions?

- Is there a connection to 2+1d integrable models?

- This is not just formal – actual experimental proposals have been made using FQHE edges.

  Clarke, Alicea, Shtengel; Lindner, Berg, Refeal, and Stern

- Does this say anything interesting about $\mathbb{Z}_n$ magnets?