Random rectangle-triangle tilings and Painlevé VI

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Based on ideas of:
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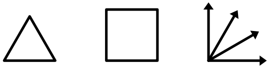
Andrei Okounkov
Paul Zinn-Justin

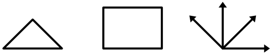
Rectangle-triangle random tilings

Random tilings of triangles and rectangles with long side ℓ.

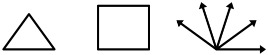
- Square Triangle
  $\ell = 1$, 12-fold.



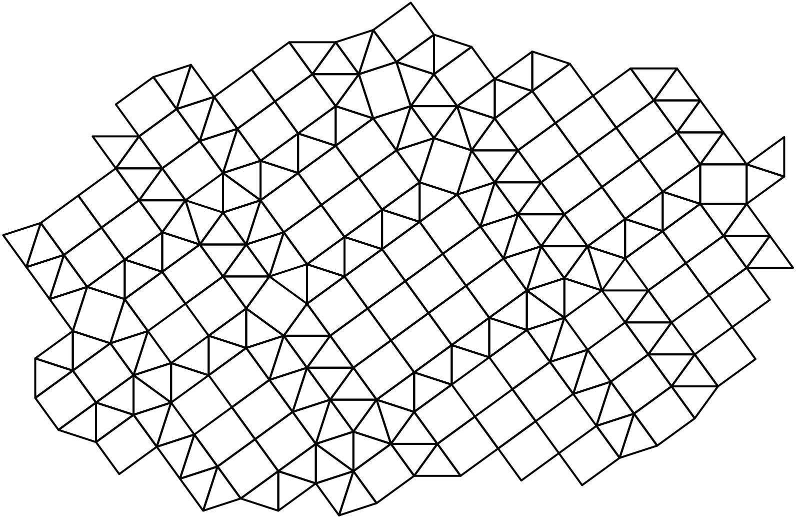
- Rectangle Triangle
  $\ell = \sqrt{2}$, 8-fold.



- Rectangle Triangle
  $\ell = 2\cos\frac{3\pi}{10}$, 10-fold.



10-fold rectangle triangle tiling



Transfer matrix T

T generates ensemble of tilings by adding a new row of tiles.

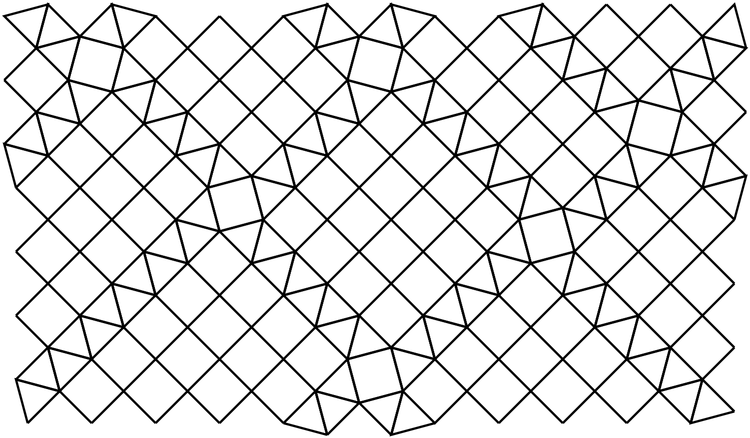
The partition sum

$$Z = \mathrm{Tr}\, T^N = \sum_i \Lambda_i^N,$$

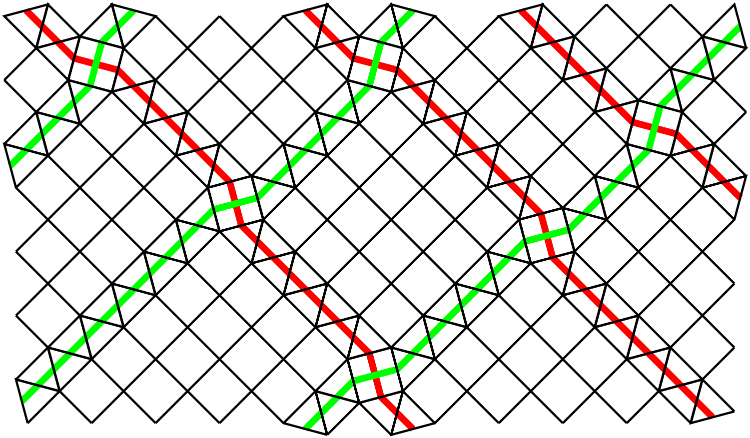
is related to the entropy

$$\sigma = \frac{1}{N} \log Z \sim \log \Lambda_{\max}$$

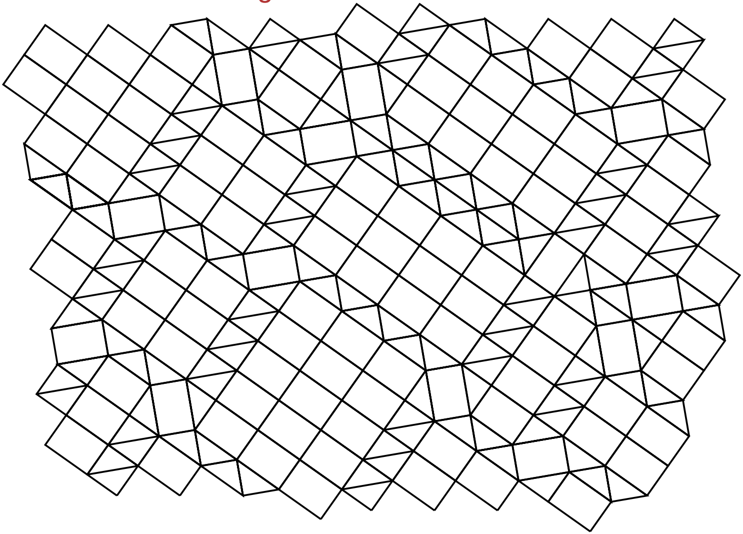
Domain walls: 12-fold tiling



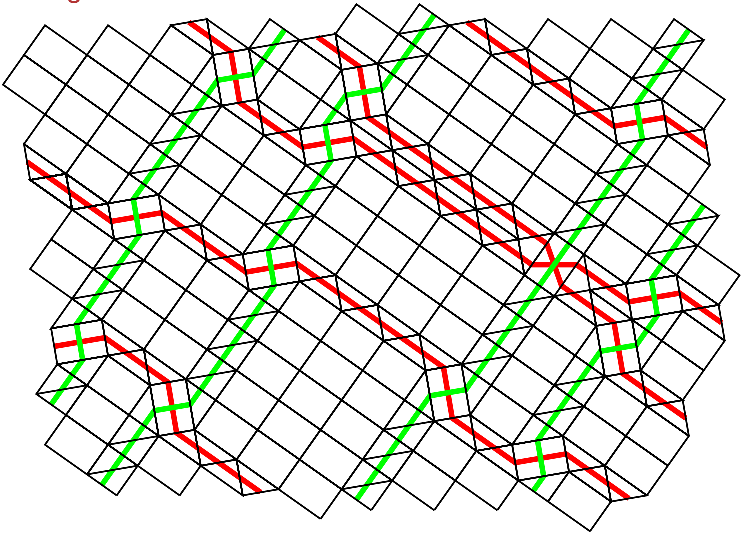
Domain walls: 12-fold tiling



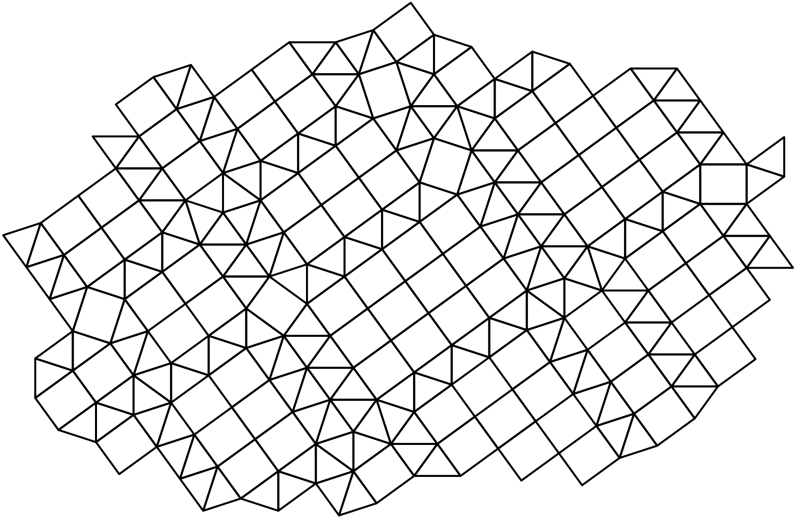
Domain walls: 8-fold tiling



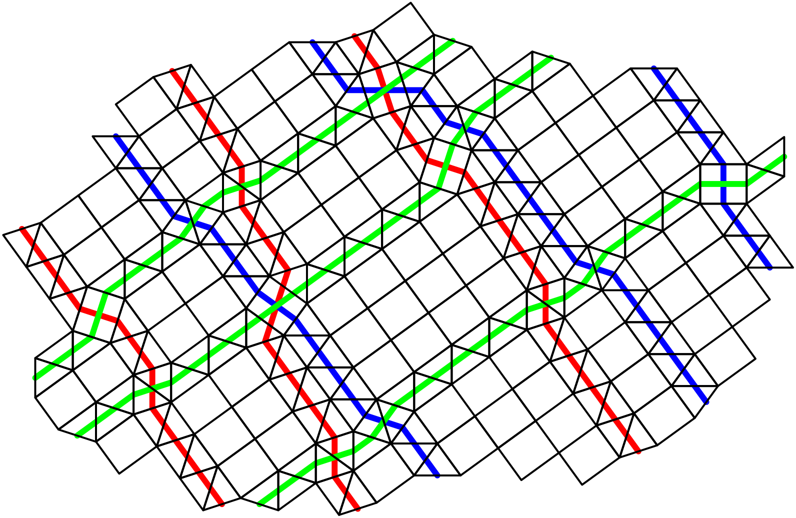
8-fold tiling



10-fold tiling



10-fold tiling



Bethe ansatz

- Domain walls are interpreted as “world lines” of particles
- The transfer matrix is the evolution operator
- It can be diagonalised using a plane wave ansatz

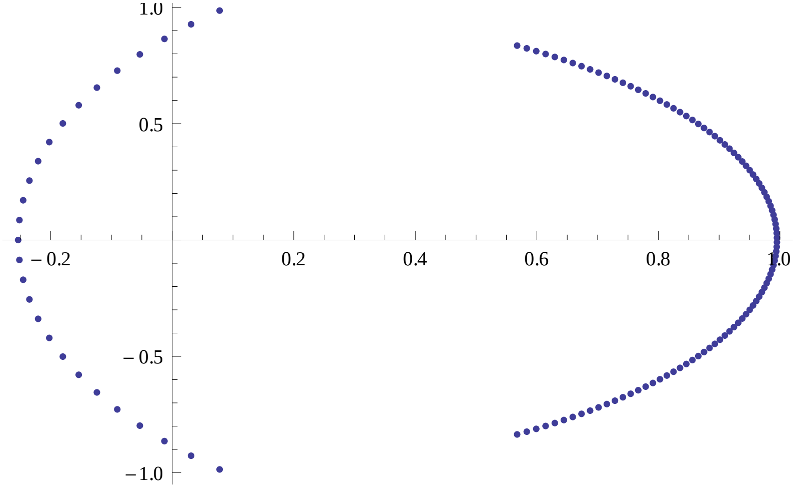
The solution of the eigenvalue problem has the following form

$$\begin{aligned} \Lambda &= \left(\prod_{i=1}^{n} u_i^{-1} + 1\right) \prod_{j=1}^{m} v_j, \\ u_i^N &= (-)^{n-1} \prod_{j=1}^{m} \frac{1}{u_i - v_j}, \\ 1 &= (-1)^{m-1} \prod_{i=1}^{n} \frac{1}{u_i - v_j}, \end{aligned}$$

Solution: 10-fold

$$
\begin{aligned}
\Lambda &= \prod_{i=1}^{n} u_i \prod_{j=1}^{m} v_j, \\
u_i^N &= (-)^{n-1} \prod_{j=1}^{m} (u_i - v_j) \prod_{l=1}^{n_2} (w_l + u_i - u_i^{-1}), \\
v_j^N &= (-)^{m-1} \prod_{i=1}^{n} (u_i - v_j), \\
(-)^{n_2} &= \prod_{i=1}^{n} (w_l + u_i - u_i^{-1}).
\end{aligned}
$$

Numerical solution: 12-fold



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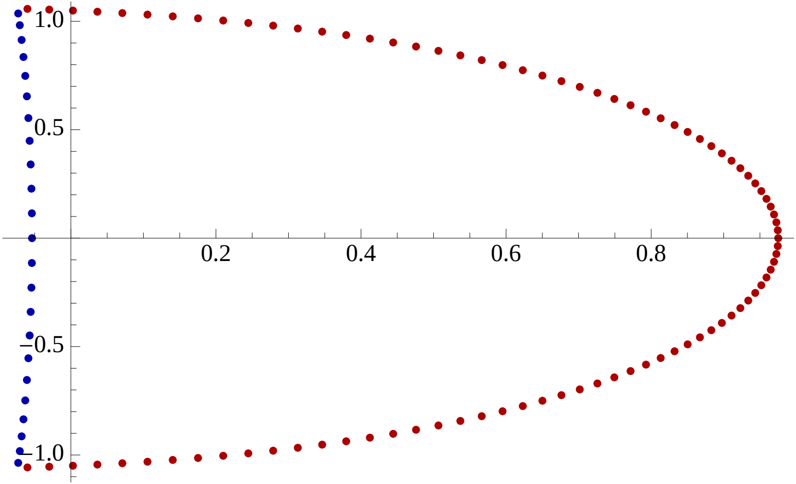

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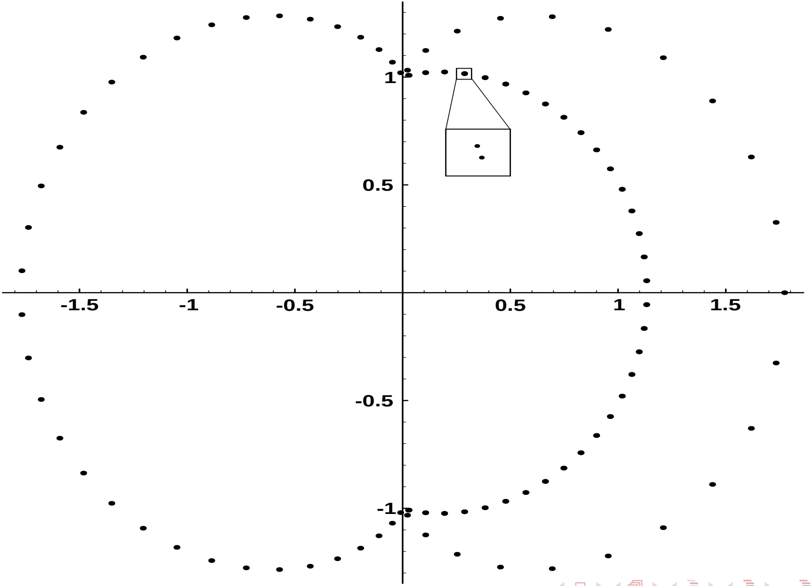

Logarithmic form

Bethe ansatz in logarithmic form:

$$F_1(u) = \log u + \frac{1}{N}\sum_{j=1}^{M_2} \log(u - v_j),$$

$$F_2(v) = \frac{1}{N}\sum_{i=1}^{M_1} \log(u_i - v).$$

$F_1(u_i) = 2\pi\mathrm{i}\mathbb{Z} \qquad F_2(v_j) = 2\pi\mathrm{i}\mathbb{Z}$ for all i and j.

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In the thermodynamic limit ($N \to \infty$) the Bethe ansatz equations imply the following integral equations for the root densities $f_i(u) := F_i'(u)$:

$$f_1(u) = \frac{1}{u} - \frac{1}{2\pi \mathrm{i}} \int_V \frac{f_2(v)}{v - u} \mathrm{d}v, \qquad f_2(v) = -\frac{1}{2\pi \mathrm{i}} \int_U \frac{f_1(u)}{u - v} \mathrm{d}u.$$

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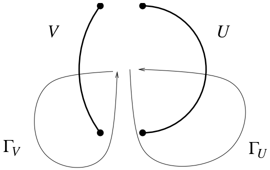
The theory of isomonodromic transformations allows us to determine the exact form of this system, which we can then hope to solve using expansion and/or perturbation techniques.

Monodromy



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$$G(z) = a_1 f_1(z) + a_2 f_2(z),$$

The analytic continuation along paths Γ_V and Γ_U crossing the curves U and V can be summarised with the following monodromy matrices:

$$\Gamma_U : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \Gamma_V : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Differential equation

Note that (f_1, f_2) satisfies a Riemann-Hilbert problem (RHP) with constant matrices

$$M_V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M_U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$Y = \begin{pmatrix} f_1 & f_2 \\ \tilde{f}_1 & \tilde{f}_2 \end{pmatrix},$$

be a matrix solution. After traversing a path γ, $Y(z)$ has changed to $Y(z_\gamma)$ and we have

$$Y(z_\gamma) = Y(z) M_\gamma,$$

where M_γ is the monodromy matrix corresponding to γ.

Differential equation

$$Y(z_\gamma) = Y(z)M_\gamma,$$

implies that

$$Y'(z_\gamma)Y(z_\gamma)^{-1} = Y'(z)Y(z)^{-1},$$

is monodromy free, and therefore meromorphic around z. So we can write

$$\frac{\partial}{\partial z}Y(z) = A(z)Y(z),$$

where $A(z)$ is analytic everywhere except possibly at the endpoints.

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We will assume that $A(z)$ has only simple poles at the endpoints:

$$A(z) = \sum_{i=1}^{4} \frac{A_i}{z - b_i}.$$

The question is: What are the matrices A_i for our problem?

Differential equation

Because infinity is monodromy free we have

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Note that due to the specific form of the monodromy, the matrices A_i will not be diagonalisable.

Isomonodromic deformations

In the general the solution $Y(z)$ of

$$\frac{\partial}{\partial z} Y(z) = A(z) Y(z),$$

will depend on all the parameters b_i, $i = 1, \ldots, 4$.

Isomonodromic deformations

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will depend on all the parameters b_i, $i = 1, \ldots, 4$.

The matrices $A_i = A_i(b)$ will have to depend on b as well, and must satisfy the following consistency conditions:

$$\frac{\partial}{\partial z} Y(z, b) = \sum_{i=1}^{4} \frac{A_i(b)}{z - b_i} Y(z, b),$$
$$\frac{\partial}{\partial b_i} Y(z, b) = -\frac{A_i(b)}{z - b_i} Y(z, b), \qquad (i = 1, \ldots, 4).$$

Painlevé VI

Equivalently, we require the Schlesinger equations

$$\frac{\partial}{\partial b_j} A_i(b) = \frac{[A_j, A_i]}{b_j - b_i} \qquad (j \neq i),$$
$$\frac{\partial}{\partial b_i} A_i(b) = -\sum_{j \neq i} \frac{[A_j, A_i]}{b_j - b_i}.$$

Sending b_1, b_2, b_3 and b_4 to $0, 1, t, \infty$, these equations imply Painlevé VI:

$$-\frac{(f-1)f(f-t)}{2(t-1)^2 t^2}\left(1 + \frac{t(t-1)}{(f-t)^2}\right) - \frac{1}{2}\dot{f}^2\left(\frac{1}{f-t} + \frac{1}{f} + \frac{1}{f-1}\right) + \dot{f}\left(\frac{1}{f-t} + \frac{1}{t} + \frac{1}{t-1}\right) + \ddot{f} = 0$$

where $f = f(A_0, A_1)$.

Conclusion

- Bethe ansatz for random tilings $\Rightarrow$ matrix Riemann-Hilbert problem
- Find isomonodromic deformation
- Leads to differential equation with parameter given by Painlevé VI.
- General solution?