Invariant Classification of second-order conformally-superintegrable Systems

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Outline

In this talk we will:

1. Discuss the concept of a conformally-superintegrable system
2. Examine the equations that govern a 3-dimensional, second-order, conformally-superintegrable system.
3. Discuss how (locally) the classification of such systems can be achieved by considering the projective invariants of a 6^{th} order polynomial (i.e. root multiplicity and cross ratios).
4. Discuss how algebraic varieties (i.e. polynomial ideals) can be used to describe global invariants of such systems.

Hamiltonian Mechanics

For an (autonomous) classical system with Hamiltonian H defined over the symplectic phase space $(\mathbf{p}, \mathbf{x})$ there is the Poisson bracket $\{\cdot, \cdot\}$ with the properties

Poisson Bracket

- $\{a, b\} = \sum_{i=1}^{n} \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial x_i}$
- $\frac{dX}{dt} = \{X, H\}$
- $\frac{dx_i}{dt} = \{x_i, H\} = \frac{\partial H}{\partial p_i}$
- $\frac{dp_i}{dt} = \{p_i, H\} = -\frac{\partial H}{\partial x_i}$

Integrability and Conformal-Integrability

- Two functions $A(\mathbf{x}, \mathbf{p}), B(\mathbf{x}, \mathbf{p})$ are said to be in involution if and only if $\{A(\mathbf{x}, \mathbf{p}), B(\mathbf{x}, \mathbf{p})\} = 0$.
- A function $L(\mathbf{x}, \mathbf{p})$ is called an integral of the Hamiltonian H if
$$\{L(\mathbf{x}, \mathbf{p}), H(\mathbf{x}, \mathbf{p})\} = 0.$$
- A function $L(\mathbf{x}, \mathbf{p})$ is called a conformal-integral of the Hamiltonian H if
$$\{L(\mathbf{x}, \mathbf{p}), H(\mathbf{x}, \mathbf{p})\} = \rho(\mathbf{x}, \mathbf{p})H(\mathbf{x}, \mathbf{p})$$
(for some function $\rho(\mathbf{x}, \mathbf{p})$).
- An n-dimensional system is said to be (conformally-)integrable if there are $n - 1$ functionally independent (conformal-)integrals in addition to the Hamiltonian.
- A system is said to be (conformally-)superintegrable if their are more than the prerequisite $n - 1$ integrals.

2nd-Order Conformal-Superintegrability in n-Dimensions

Our problem

- Maximally superintegrable, i.e. $2n-1$ independent conformal-integrals.
- Flat space, i.e. over a space with Riemmanian metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2,$$

  with the Hamiltonian

$$H = \sum_{i=1}^{n} p_i^2 + V(\mathbf{x})$$

- Second-Order, i.e. every conformal-integral is quadratic in the momenta

$$L = \sum_{i,j=1}^{n} a^{ij}(\mathbf{x}) p_i p_j + W(\mathbf{x}), \qquad a^{ij}(\mathbf{x}) = a^{ji}(\mathbf{x}).$$

- Maximum Parameter, potentials depending on 5 parameters.

Smorodinsky-Winternitz Potential

Potential V_I, known as the Smorodinsky-Winternitz Potential

$$H = (p_x^2 + p_y^2 + p_z^2) + a(x^2 + y^2 + z^2) + \frac{b}{x^2} + \frac{c}{y^2} + \frac{d}{z^2} + e,$$

Which has 6 linearly independent (but functionally dependent) second-order symmetries

$$\begin{aligned}
L_1 &= p_x^2 + ax^2 + \frac{b}{x^2}, \\
L_2 &= p_y^2 + ay^2 + \frac{c}{y^2}, \\
L_3 &= p_z^2 + az^2 + \frac{d}{z^2}, \\
L_{12} &= (p_x y - p_y x)^2 + b\frac{y^2}{x^2} + c\frac{x^2}{y^2}, \\
L_{23} &= (p_y z - p_z y)^2 + c\frac{z^2}{y^2} + d\frac{y^2}{z^2}, \\
L_{13} &= (p_x z - p_z x)^2 + b\frac{z^2}{x^2} + d\frac{x^2}{z^2}.
\end{aligned}$$

Maximum-Parameter Potentials

It can be shown that any 2nd-order system with functionally linearly independent constants satisfies a PDE of the form

$$
\begin{aligned}
\frac{\partial^2 V}{\partial x_2^2} &= \frac{\partial^2 V}{\partial x_1^2} + A^{22}(\mathbf{x})\frac{\partial V}{\partial x_1} + B^{22}(\mathbf{x})\frac{\partial V}{\partial x_1} + C^{22}(\mathbf{x})\frac{\partial V}{\partial x_3} + D^{22}(\mathbf{x})V,\\
\frac{\partial^2 V}{\partial x_3^2} &= \frac{\partial^2 V}{\partial x_1^2} + A^{33}(\mathbf{x})\frac{\partial V}{\partial x_1} + B^{33}(\mathbf{x})\frac{\partial V}{\partial x_1} + C^{33}(\mathbf{x})\frac{\partial V}{\partial x_3} + D^{33}(\mathbf{x})V,\\
\frac{\partial^2 V}{\partial x_1 \partial x_2} &= A^{12}(\mathbf{x})\frac{\partial V}{\partial x_1} + B^{12}(\mathbf{x})\frac{\partial V}{\partial x_2} + C^{12}(\mathbf{x})\frac{\partial V}{\partial x_3} + D^{12}(\mathbf{x})V,\\
\frac{\partial^2 V}{\partial x_1 \partial x_3} &= A^{13}(\mathbf{x})\frac{\partial V}{\partial x_1} + B^{13}(\mathbf{x})\frac{\partial V}{\partial x_2} + C^{13}(\mathbf{x})\frac{\partial V}{\partial x_3} + D^{13}(\mathbf{x})V,\\
\frac{\partial^2 V}{\partial x_2 \partial x_3} &= A^{23}(\mathbf{x})\frac{\partial V}{\partial x_1} + B^{23}(\mathbf{x})\frac{\partial V}{\partial x_2} + C^{23}(\mathbf{x})\frac{\partial V}{\partial x_3} + D^{23}(\mathbf{x})V,
\end{aligned}
$$

for 20 functions $A^{ij}(\mathbf{x}), B^{ij}(\mathbf{x}), C^{ij}(\mathbf{x}), D^{ij}(\mathbf{x})$.

List of Known Potentials

$$r^2 = x^2 + y^2 + z^2$$

SW	$\dfrac{a}{(1+r^2)^2} + \dfrac{b}{x^2} + \dfrac{c}{y^2} + \dfrac{d}{z^2} + \dfrac{e}{(-1+r^2)^2}$
I	$ar^2 + \dfrac{b}{x^2} + \dfrac{c}{y^2} + \dfrac{d}{z^2} + e$
II	$ar^2 + b\dfrac{(x-iy)}{(x+iy)^3} + c\dfrac{1}{(x+iy)^2} + d\dfrac{1}{z^2} + e$
III	$ar^2 + b\dfrac{1}{(x+iy)^2} + c\dfrac{z}{(x+iy)^3} + d\dfrac{x^2+y^2-3z^2}{(x+iy)^4} + e$
IV	$a\left(4x^2+y^2+z^2\right) + bx + \dfrac{c}{y^2} + \dfrac{d}{z^2} + e$
V	$a\left(4z^2+x^2+y^2\right) + bz + \dfrac{c}{(x+iy)^2} + d\dfrac{x-iy}{(x+iy)^3} + e$
VI	$a\left(z^2 - 2(x-iy)^3 + 4(x^2+y^2)\right) + b(2x+2iy-3(x-iy)^2) + c(x-iy) + \dfrac{d}{z^2} + e$
VII	$a\,(x+iy) + b(3(x+iy)^2+z) + c(16(x+iy)^3 + (x-iy) + 12z(x+iy)) + d(5(x+iy)^4 + r^2 + 6(x+iy)^2 z) + e$
O	$ar^2 + bx + cy + dz + e$
OO	$a\left(4x^2+4y^2+z^2\right) + bx + cy + \dfrac{d}{z^2} + e$
A	$a\left((x-iy)^3 + 6(x^2+y^2+z^2)\right) + b\left((x-iy)^2 + 2(x+iy)\right) + c(x-iy) + dz + e$

Classifying Features of V_{VI}

The form of the PDE above naturally gives an action of conformal group on the coefficient functions. If we consider the rotation representations we naturally get a $\mathfrak{su}(2) \cong \mathfrak{so}(3, \mathbb{C})$ representation via the polynomial

$$\begin{aligned} p(\eta) = & -1/8\, i \left(2\, iB^{11} + 4\, iA^{12} - 2\, iB^{22} - 3\, A^{11} + 5\, B^{12} - C^{13}\right) \\ & + 3/2\, i \left(2\, iC^{12} - C^{11} + C^{22}\right) \eta \\ & + 3/8\, i \left(14\, iB^{11} - 4\, iA^{12} + 10\, iB^{22} - 3\, A^{11} - 3\, B^{12} + 7\, C^{13}\right) \eta^2 \\ & + i \left(C^{11} + 4\, A^{13} + 5\, C^{22}\right) \eta^3 \\ & + 3/8\, i \left(14\, iB^{11} - 4\, iA^{12} + 10\, iB^{22} + 3\, A^{11} + 3\, B^{12} - 7\, C^{13}\right) \eta^4 \\ & - 3/2\, i \left(2\, iC^{12} + C^{11} - C^{22}\right) \eta^5 \\ & - 1/8\, i \left(2\, iB^{11} + 4\, iA^{12} + 3\, A^{11} - 5\, B^{12} + C^{13} - 2\, iB^{22}\right) \eta^6 \end{aligned}$$

for potential VI, in the form above, this becomes

$$p(\eta) = 3i(\frac{2}{z} + \eta^3)\eta^3$$

which has the 4 roots

$$\eta = 0, \sqrt[3]{\frac{-2}{z}} e^{i\frac{2k\pi}{3}}; \qquad k = 1, 2, 3.$$

Classifying Features of V_{VI}

If we calculate the so called multiratio[1] of the four roots via

$$CR(r_1, r_2, r_3, r_4) = \frac{(r_1 - r_2)(r_3 - r_4)}{(r_2 - r_3)(r_4 - r_1)}$$

we find

$$CR\left(0, \sqrt[3]{\frac{-2}{z}}e^{i\frac{2\pi}{3}}, \sqrt[3]{\frac{-2}{z}}e^{i\frac{4\pi}{3}}, \sqrt[3]{\frac{-2}{z}}\right) = e^{\frac{-i\pi}{3}}$$

[1] equivalent to the better known cross ratio

Classifying Features of V_{VI}

Performing the conformal change of variables

$$x = \frac{u}{u^2+v^2+w^2}, \quad y = \frac{v}{u^2+v^2+w^2}, \quad z = \frac{w}{u^2+v^2+w^2},$$

we can re-derive the coefficient functions and resubstitute these into the $\mathfrak{so}(3,\mathbb{C})$ representation to obtain

$$\widehat{p}(\eta) = \frac{-3\,i\,(\eta w - (u+iv))^3}{w\,(u^2+v^2+w^2)^4} \cdot \left(\begin{array}{c} \left(-w^4 + 2\,(iv+u)^3\,w^3 + 2\,(u^2+v^3)\,(iv+u)^3\right)\eta^3 \\ + \left(6\,(iv+u)^2\,w^4 + (-3\,iv + 3\,u)\,w^3 + 6\,(u^2+v^3)\,(iv+u)^2\,w\right)\eta^2 \\ + \left((6\,iv + 6\,u)\,w^5 + \left(6\,ivu^2 + 6\,ivu + 3\,v^2 + 6\,u^3 + 6\,iv^4 + 6\,v^3 u - 3\,u^2\right)w^2\right)\eta \\ +2\,w^6 + \left(2\,u^2 + 2\,v^3\right)w^3 - (-u+iv)^3\,w \end{array} \right)$$

Whose roots are

$$\eta = \frac{u-iv}{w}, \frac{-w\gamma_k + u - iv}{(u+iv)\gamma_k + w}; \qquad \gamma_k = \sqrt[3]{\frac{-2(u^2+v^2+w^2)}{w}}\, e^{i\frac{2k\pi}{3}}$$

Classifying Features of V_{VI}

In both presentations of potential VI the polynomial $p(\eta)$ had a root of multiplicity 3 and three roots of multiplicity 1. These roots were related by the explicit formula

$$r \to \frac{-wr + (u - iv)}{(u + iv)r + w}$$

and thus, being a projective transformation, preserves the multiratio

$$CR\left(\frac{u - iv}{w}, \frac{-w\gamma_1 + u - iv}{(u + iv)\gamma_1 + w}, \frac{-w\gamma_2 + u - iv}{(u + iv)\gamma_2 + w}, \frac{-w\gamma_3 + u - iv}{(u + iv)\gamma_3 + w}\right) = e^{\frac{-i\pi}{3}}.$$

Examining root multiplicities and the behaviour of the multiratios is all that will be required to classify these second-order conformally-superintegrable systems.

Rotations Representations

Lie derivatives corresponding to rotations act linearly on the A, B, C's.

We can choose a basis which splits into blocks of 3 and 7 dimensions on which the rotations act irreducibly.

$$\begin{aligned}
X_{+1} &= \tfrac{1}{10}\left(2\,iB^{11} - 2\,iA^{12} - A^{11} - B^{12} - C^{13}\right),\\
X_0 &= \tfrac{-1}{5}\sqrt{2}\left(C^{11} - A^{13}\right),\\
X_{-1} &= \tfrac{1}{10}\left(2\,iB^{11} - 2\,iA^{12} + A^{11} + B^{12} + C^{13}\right),
\end{aligned}$$

$$\begin{aligned}
Y_{+3} &= \tfrac{-1}{8}\,i\left(2\,iB^{11} + 4\,iA^{12} + 3\,A^{11} - 5\,B^{12} + C^{13} - 2\,iB^{22}\right),\\
Y_{+2} &= \tfrac{1}{4}\,i\sqrt{6}\left(2\,iC^{12} + C^{11} - C^{22}\right),\\
Y_{+1} &= \tfrac{1}{40}\,i\sqrt{15}\left(14\,iB^{11} - 4\,iA^{12} + 10\,iB^{22} + 3\,A^{11} + 3\,B^{12} - 7\,C^{13}\right),\\
Y_0 &= \tfrac{-1}{10}\,i\sqrt{5}\left(C^{11} + 4\,A^{13} + 5\,C^{22}\right),\\
Y_{-1} &= \tfrac{1}{40}\,i\sqrt{15}\left(14\,iB^{11} - 4\,iA^{12} + 10\,iB^{22} - 3\,A^{11} - 3\,B^{12} + 7\,C^{13}\right),\\
Y_{-2} &= \tfrac{-1}{4}\,i\sqrt{6}\left(2\,iC^{12} - C^{11} + C^{22}\right),\\
Y_{-3} &= \tfrac{-1}{8}\,i\left(2\,iB^{11} + 4\,iA^{12} - 2\,iB^{22} - 3\,A^{11} + 5\,B^{12} - C^{13}\right).
\end{aligned}$$

It can be shown that the conformal group acts transitively on the X's. Thus all we need to consider is the values of the Y_i's.

Maximum-Parameter Potentials

The compatibility conditions also allows us to calculate the derivatives of the 10 coefficient functions, e.g.

$$\frac{\partial Y_{+3}}{\partial z} = \frac{\sqrt{2}}{2} X_0\, Y_{+3} + \frac{\sqrt{6}}{2} Y_{+2}\, X_{+1} + \frac{i\sqrt{5}}{15} Y_{+3}\, Y_0 - \frac{i\,\sqrt{10}}{45} Y_{+1}\, Y_{+2},$$

$$\begin{aligned}\frac{\partial X_{+1}}{\partial z} &= \frac{3\sqrt{2}}{2} X_0 X_{+1} - \frac{2\sqrt{10}}{45} Y_{+2}\, Y_{-1} + \frac{4\sqrt{3}}{135} Y_0\, Y_{+1} \\ &\quad + \frac{2\sqrt{6}}{27}\, Y_{-2}\, Y_{+3} - \frac{4i\sqrt{30}}{45} X_0\, Y_{+1} \\ &\quad + \frac{2i\sqrt{6}}{9} Y_{+2}\, X_{-1} + \frac{2i\sqrt{5}}{15} Y_0\, X_{+1}.\end{aligned}$$

These equations are integrable with no further restrictions on the values of X_i, Y_i. So the values of X_i, Y_i can be arbitrarily specified at a point.

Understanding the derivatives

We define $\partial_\pm = i\partial_y \pm \partial_x$ and $\partial_0 = \partial_z$. And then define

$$\begin{aligned}
\hat{\partial}_+(Y_m) &= \partial_+(Y_m) + X_{+1}D(Y_m) - \sum_{i=1}^{3} J_i(X_{+1})J_i(Y_m) \\
\hat{\partial}_0(Y_m) &= \partial_0(Y_m) + X_0 D(Y_m) - \sum_{i=1}^{3} J_i(X_0)J_i(Y_m) \\
\hat{\partial}_-(Y_m) &= \partial_-(Y_m) + X_{-1}D(Y_m) - \sum_{i=1}^{3} J_i(X_{-1})J_i(Y_m).
\end{aligned}$$

and interestingly $\hat{\partial}_\pm(Y_m)$ and $\hat{\partial}_0(Y_m)$ are expressions quadratic in the Y's alone. For example,

$$\begin{aligned}
\hat{\partial}_+(Y_{+3}) &= \frac{2i}{9}(Y_{+2})^2 - \frac{4i}{45\sqrt{15}} Y_{+1}Y_{+3} \\
\hat{\partial}_0(Y_{+3}) &= \frac{i\sqrt{5}}{15} Y_{+3}Y_0 - \frac{i\sqrt{10}}{45} Y_{+1}Y_{+2} \\
\hat{\partial}_-(Y_{+3}) &= -\frac{2i\sqrt{15}}{5} Y_{-1}Y_{+3} + \frac{17i\sqrt{30}}{45} Y_{+2}Y_0 - \frac{10i}{9}(Y_{+1})^2
\end{aligned}$$

Invariants of 7-dimensional representation

We wish to classify the orbits of the conformal group acting on the 7-dimensional representation carried by the Y_m.

To find invariants, it is convenient to consider the Y_m as coefficients in a polynomial,

$$p(\eta) = \sum_{k=0}^{n} a_k \eta^k, \qquad a_k = (-1)^k \sqrt{\binom{6}{k}}\ Y_{k-3}$$

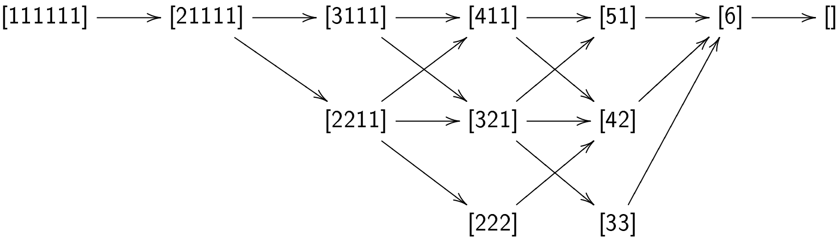
and act on this with fractional linear transformation. For example rotations take the form $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,\mathbb{C})$ and

$$p(\eta) \mapsto (c\eta + d)^n p\left(\frac{a\eta + b}{c\eta + d}\right).$$

For action of the full conformal group excluding translations of the regular point we simply take $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2,\mathbb{C})$.

Multiplicities of Roots

Since every configuration of 6 complex point is admissible, so a useful starting point would be considering the multiplicities of the roots. These have a natural hierarchy as seen below



It is well known that having a repeated root correspond to the vanishing of the discriminant (10^{th} order in this case). The other conditions are not so well known but ideals can be routinely generated using most symbolic algebra package.

Trivial case ⇒ oscillator

The simplest case is $Y_m = 0$ for $m = -3, -2, \ldots, 3$.

This is clearly preserved by conformal motions since the derivatives of the Y's are all of forms like

$$\frac{\partial Y_{+3}}{\partial z} = \frac{\sqrt{2}}{2} X_0 \, Y_{+3} + \frac{\sqrt{6}}{2} Y_{+2} \, X_{+1} + \frac{i\sqrt{5}}{15} Y_{+3} \, Y_0 - \frac{i\sqrt{10}}{45} Y_{+1} \, Y_{+2}$$

$$\frac{\partial Y_{+3}}{\partial y} = \frac{-\sqrt{15}}{5} Y_{+3} \, Y_{-1} + \frac{17}{90}\sqrt{30} \, Y_{+2} \, Y_0 - \frac{5}{9} {Y_{+1}}^2 - \frac{2\sqrt{15}}{45} Y_{+3} \, Y_{+1} + \frac{1}{9} {Y_{+2}}^2 + \frac{i\sqrt{3}}{2} X_0 \, Y_{+2} - i \left(X_{-1} - 2\, X_{+1}\right) Y_{+3}$$

Using the transitivity of the conformal group, we can also set $X_{\pm} = X_0 = 0$ and then solving we find the harmonic oscillator with linear terms

$$V_O = \alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z.$$

Closure of ideal

For a polynomial ideal I, denote the process of closing differentially and determining the radical by $\bar{I}$, that is

$$\bar{I} = \sqrt{< I,\ \partial I,\ \partial^2 I, \dots >}$$

Root multiplicities [6], [51], [411], [33] and [3111] are differentially closed.

Root multiplicities [42], [321], [222], [2211] and [21111] are not stable under translation.

$$\overline{I_{[42]}} = I_{[6]}, \qquad \overline{I_{[321]}} = I_{[33]} \cup I_{[51]}, \qquad \overline{I_{[222]}} = I_{[6]},$$
$$\overline{I_{[2211]}} = I_{[411]} \cup I_{[33]}, \qquad \overline{I_{[21111]}} = I_{[3111]}.$$

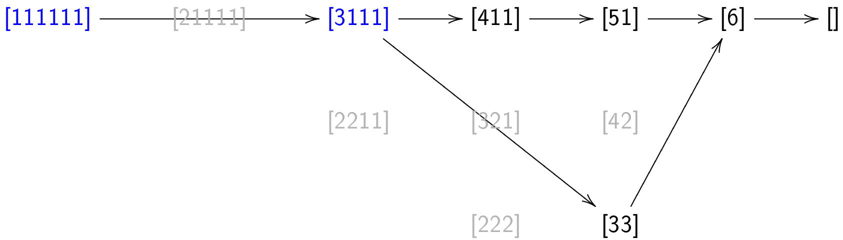
The root multiplicities partially split the known systems.

[]	[6]	[51]	[411]	[33]	[3111]	[111111]
O	*A*	*VII*	*III*, *V*	*OO*	*II*, *VI*	*S*, *I*, *IV*

Systems *III* and *V* can not be distinguished here because they are Stäckel equivalent.

Multiplicities of Roots

So, pruning our diagram, we have



The conformal group act transitive on 3-tuples of points So most of these cases the analysis is now complete. but for cases [3111] and [111111] there is still work to be done.

Cross ratio

If there are 4 roots we can define their cross ratio

$$\lambda = (z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

It is invariant under all conformal transformations except translations.

Except in special cases, permuting roots gives 6 distinct cross ratios

$$\lambda, \quad \frac{1}{\lambda}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda-1} \quad \text{and} \quad \frac{\lambda-1}{\lambda}.$$

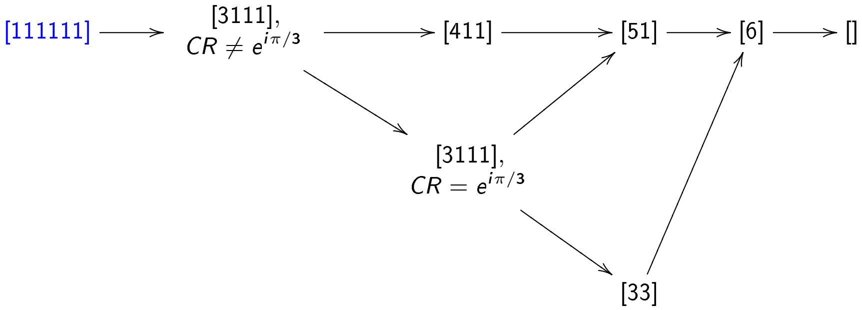
For $p(z) = z^3(z^3 - 1)$, only two distinct cross ratios of the roots exist, $e^{\pm \pi i/3}$.

The condition that the cross ratio has this value is stable under translations and splits the [3111] case with systems *II* and *VI*. System *VI* has cross ratio $e^{\pm \pi i/3}$.

For greater than 4 roots more independent cross ratios exist and they can be used to construct invariants that split the [111111] cases.

Multiplicities of Roots

Updating our diagram we now have the



Conclusions and Future Directions

For the final case, [111111], we can demonstrate a 6^{th} order ideal that separates off potential IV, and a 10^{th} order ideal that splits off potential I. The remaining potential, SW, (the generic spherical potential) corresponds to the bulk of the $\mathbb{C}^{10}$ space.

To close the problem all that is left to do is:

- Show the potentials SW, I, IV are the only solutions with a [111111] roots structure.
- Perform an analogous treatment of the 4 dimensional case, exploiting the $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ representations.
- Use similar techniques to examine the 3 parameter potentials in 3d space.