Osculating Lattice Paths and the Bethe Ansatz

ANZAMP 2012

R Brak and W Galleaus

Department of Mathematics and Statistics
University of Melbourne

December 2, 2012

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

Definition

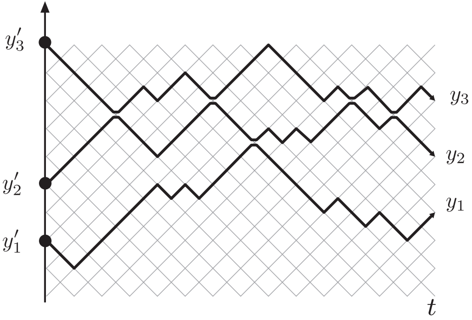
Definition: Osculating lattice paths

- An N-tuple of binomial paths: $(b_1, \ldots, b_N)$
- Non-crossing
- Shared vertices – "osculations"
- No shared edges

Definition

Definition: Osculating lattice paths

- An N-tuple of binomial paths: $(b_1, \ldots, b_N)$
- Non-crossing
- Shared vertices – "osculations"
- No shared edges
- Example: Watermelon geometry:



Motivation

Combinatorial Understanding of Product forms: Alternating Sign Matrices

- $N \times N$ matrices A_{ij}
- $A_{ij} \in \{-1, 0, 1\}$
- $\sum_i A_{ij} = 1$
- $\sum_j A_{ij} = 1$
- Alternate in sign and $+1$ first

Motivation

Combinatorial Understanding of Product forms: Alternating Sign Matrices

- $N \times N$ matrices A_{ij}
- $A_{ij} \in \{-1, 0, 1\}$
- $\sum_i A_{ij} = 1$
- $\sum_j A_{ij} = 1$
- Alternate in sign and $+1$ first
- Example $N = 3$:

$$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & -1 & 1 \\ \cdot & 1 & \cdot \end{pmatrix}$$

$$\text{Product: Number of } N \times N \text{ matrices} = \prod_{i=1}^{N-1} \frac{(3i+1)!}{(n+i)!}$$

Bijection

- ASM biject to Osculating paths in “corner” geometry

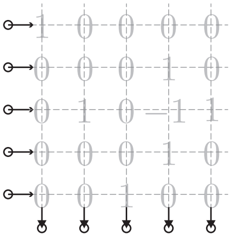
Bijection

- ASM biject to Osculating paths in “corner” geometry

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}$$

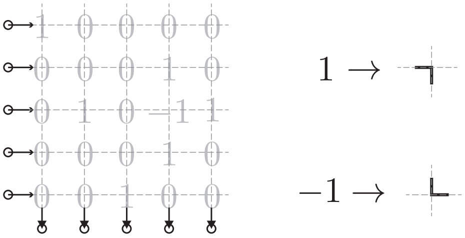
Bijection

- ASM biject to Osculating paths in "corner" geometry



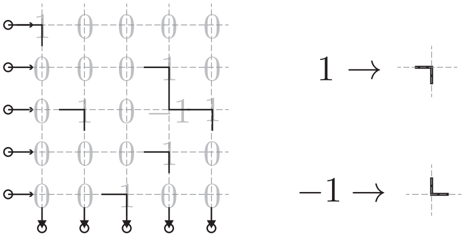
Bijection

- ASM biject to Osculating paths in “corner” geometry



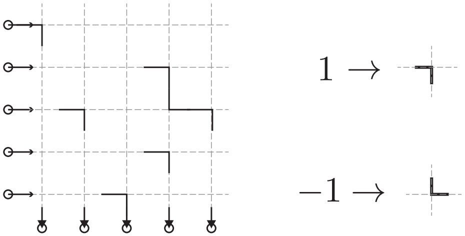
Bijection

- ASM biject to Osculating paths in “corner” geometry



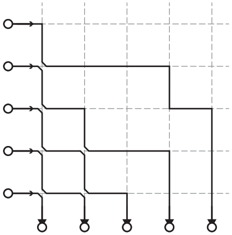
Bijection

- ASM biject to Osculating paths in “corner” geometry

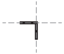


Bijection

- ASM biject to Osculating paths in "corner" geometry



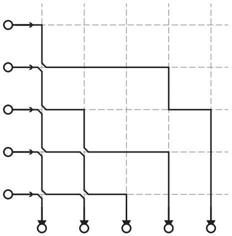
$$1 \rightarrow$$

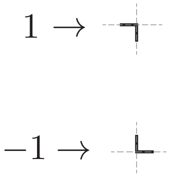


$$-1 \rightarrow$$

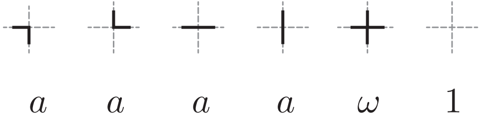
Bijection

- ASM biject to Osculating paths in “corner” geometry



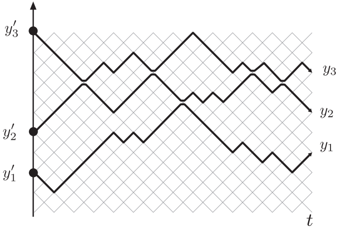


- Same as a 6-vertex model



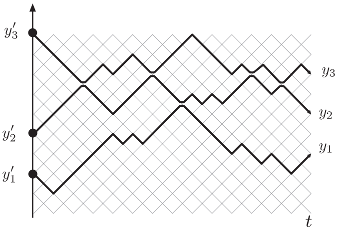
Back to “Watermelons”

- Back to ‘Watermelons’:



Back to "Watermelons"

- Back to 'Watermelons':



- Use column-transfer matrix equation:

$$Z(\mathbf{y}) = \mathbf{T}^t Z(\mathbf{y}')$$

- Initial: $\mathbf{y} = y_1' < y_2' < \cdots < y_N'$, same parity.
- Final: $\mathbf{y} = y_1 \leqslant y_2 \leqslant \cdots \leqslant y_N$.

Partial Difference Equations

$$Z(\mathbf{y}; t+1) = \mathbf{T}\, Z(\mathbf{y}; t)$$

Split into $1 + F_n$ cases:

- Bulk: $y_1 < y_2 < \cdots < y_N$

$$Z(\mathbf{y}; t+1) = \sum_{e_1 \in \{\pm 1\}} \cdots \sum_{e_N \in \{\pm 1\}} Z(\mathbf{y} + \mathbf{e}; t)$$

Partial Difference Equations

$$Z(\mathbf{y}; t+1) = \mathbf{T}\, Z(\mathbf{y}; t)$$

Split into $1 + F_n$ cases:

- Bulk: $y_1 < y_2 < \cdots < y_N$

$$Z(\mathbf{y}; t+1) = \sum_{e_1 \in \{\pm 1\}} \cdots \sum_{e_N \in \{\pm 1\}} Z(\mathbf{y} + \mathbf{e}; t)$$

- Osculations: One equation for each possible osculation case.

Partial Difference Equations

$$Z(\mathbf{y}; t+1) = \mathbf{T}\, Z(\mathbf{y}; t)$$

Split into $1 + F_n$ cases:

- Bulk: $y_1 < y_2 < \cdots < y_N$

$$Z(\mathbf{y}; t+1) = \sum_{e_1 \in \{\pm 1\}} \cdots \sum_{e_N \in \{\pm 1\}} Z(\mathbf{y} + \mathbf{e}; t)$$

- Osculations: One equation for each possible osculation case.
- Example $N = 3$

$$\begin{aligned} Z(\mathbf{y}; t+1) =& \omega\, Z(y_1, y_1, y_3; t), \qquad y_1 = y_2 < y_3 \\ Z(\mathbf{y}; t+1) =& \omega\, Z(y_1, y_2, y_2; t), \qquad y_1 < y_2 = y_3 \end{aligned}$$

Partial Difference Equations

$$Z(\mathbf{y}; t+1) = \mathbf{T}\, Z(\mathbf{y}; t)$$

Split into $1 + F_n$ cases:

- Bulk: $y_1 < y_2 < \cdots < y_N$

$$Z(\mathbf{y}; t+1) = \sum_{e_1 \in \{\pm 1\}} \cdots \sum_{e_N \in \{\pm 1\}} Z(\mathbf{y} + \mathbf{e}; t)$$

- Osculations: One equation for each possible osculation case.
- Example $N = 3$

$$\begin{aligned} Z(\mathbf{y}; t+1) =& \omega\, Z(y_1, y_1, y_3; t), \qquad y_1 = y_2 < y_3 \\ Z(\mathbf{y}; t+1) =& \omega\, Z(y_1, y_2, y_2; t), \qquad y_1 < y_2 = y_3 \end{aligned}$$

- Initial Condition:

$$Z(\mathbf{y}; 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$$

with $y_1 < y_2 < \cdots < y_N$ and $y_1' < y_2' < \cdots < y_N'$.

Solving

- $Z(\mathbf{y}; t) \in \mathbb{Z}[\omega]$ – a sequence of integer coefficient polynomials in ω.

Solving

- $Z(\mathbf{y}; t) \in \mathbb{Z}[\omega]$ – a sequence of integer coefficient polynomials in ω.

Solving

- $Z(\mathbf{y}; t) \in \mathbb{Z}[\omega]$ – a sequence of integer coefficient polynomials in ω.
- Bulk:

$$Z^B(\mathbf{x}; \mathbf{y}, t) = \prod_{i=1}^{N} x_i^{y_i} \left(x_i + \frac{1}{x_i} \right)^t$$

Solving

- $Z(\mathbf{y}; t) \in \mathbb{Z}[\omega]$ – a sequence of integer coefficient polynomials in ω.
- Bulk:

$$Z^B(\mathbf{x}; \mathbf{y}, t) = \prod_{i=1}^{N} x_i^{y_i} \left(x_i + \frac{1}{x_i} \right)^t$$

- $Z^B(\mathbf{x}; \mathbf{y}, t) \in \mathbb{Z}[[\mathbf{x}]]$ where $\mathbf{x} = x_1, \ldots x_N$

Solving

- $Z(\mathbf{y}; t) \in \mathbb{Z}[\omega]$ – a sequence of integer coefficient polynomials in ω.
- Bulk:

$$Z^B(\mathbf{x}; \mathbf{y}, t) = \prod_{i=1}^{N} x_i^{y_i} \left(x_i + \frac{1}{x_i} \right)^t$$

- $Z^B(\mathbf{x}; \mathbf{y}, t) \in \mathbb{Z}[[\mathbf{x}]]$ where $\mathbf{x} = x_1, \dots x_N$
- For any permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_N \in S_N$:

$$Z^B(x_{\sigma_1}, \dots x_{\sigma_N}; \mathbf{y}, t)$$

is also a solution.

Solving

- $Z(\mathbf{y}; t) \in \mathbb{Z}[\omega]$ – a sequence of integer coefficient polynomials in ω.
- Bulk:
$$Z^B(\mathbf{x}; \mathbf{y}, t) = \prod_{i=1}^{N} x_i^{y_i} \left(x_i + \frac{1}{x_i} \right)^t$$
- $Z^B(\mathbf{x}; \mathbf{y}, t) \in \mathbb{Z}[[\mathbf{x}]]$ where $\mathbf{x} = x_1, \dots x_N$
- For any permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_N \in S_N$:
$$Z^B(x_{\sigma_1}, \dots x_{\sigma_N}; \mathbf{y}, t)$$
is also a solution.
- $N!$ solutions to try solve osculating equations

Osculating Equations

- Osculating: (Coordinate Bethe Ansatz)

$$Z^O(\mathbf{x};\mathbf{y},t) = \left(x_i + \frac{1}{x_i}\right)^t \sum_{\sigma \in S_N} A_\sigma(\omega;\mathbf{x})\, Z^B(\mathbf{x}_\sigma;\mathbf{y})$$

where $\mathbf{x}_\sigma = x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}$.

Osculating Equations

- Osculating: (Coordinate Bethe Ansatz)

$$Z^O(\mathbf{x};\mathbf{y},t) = \left(x_i + \frac{1}{x_i}\right)^t \sum_{\sigma\in S_N} A_\sigma(\omega;\mathbf{x})\, Z^B(\mathbf{x}_\sigma;\mathbf{y})$$

where $\mathbf{x}_\sigma = x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_N}$.

- Solved by

$$A_\sigma = \prod_{(i,j)\in I_\sigma} -\frac{\lambda_i\lambda_j - \omega x_j/x_i}{\lambda_i\lambda_j - \omega x_i/x_j}, \qquad \lambda_i = x_i + \frac{1}{x_i}$$

where I_σ is the set of inversions of σ.

Osculating Equations

- Osculating: (Coordinate Bethe Ansatz)

$$Z^O(\mathbf{x};\mathbf{y},t) = \left(x_i + \frac{1}{x_i}\right)^t \sum_{\sigma\in S_N} A_\sigma(\omega;\mathbf{x})\, Z^B(\mathbf{x}_\sigma;\mathbf{y})$$

  where $\mathbf{x}_\sigma = x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}$.

- Solved by

$$A_\sigma = \prod_{(i,j)\in I_\sigma} -\frac{\lambda_i\lambda_j - \omega x_j/x_i}{\lambda_i\lambda_j - \omega x_i/x_j}, \qquad \lambda_i = x_i + \frac{1}{x_i}$$

  where I_σ is the set of inversions of σ.

- Solution is now a rational function $Z^O \in \mathbb{Z}((\mathbf{x}))$.

Osculating Equations

- Osculating: (Coordinate Bethe Ansatz)

$$Z^{O}(\mathbf{x};\mathbf{y},t) = \left(x_i + \frac{1}{x_i}\right)^t \sum_{\sigma \in S_N} A_\sigma(\omega;\mathbf{x})\, Z^{B}(\mathbf{x}_\sigma;\mathbf{y})$$

  where $\mathbf{x}_\sigma = x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}$.

- Solved by

$$A_\sigma = \prod_{(i,j)\in I_\sigma} -\frac{\lambda_i\lambda_j - \omega x_j/x_i}{\lambda_i\lambda_j - \omega x_i/x_j}, \qquad \lambda_i = x_i + \frac{1}{x_i}$$

  where I_σ is the set of inversions of σ.

- Solution is now a rational function $Z^{O} \in \mathbb{Z}((\mathbf{x}))$.
- Finally, the initial condition...

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$
- Need to:

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$
- Need to:
  - 'Get rid' of $\mathbf{x}$

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$
- Need to:
  - 'Get rid' of $\mathbf{x}$
  - Add back initial heights $\mathbf{y}'$

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$
- Need to:
  - 'Get rid' of $\mathbf{x}$
  - Add back initial heights $\mathbf{y}'$
- Idea: 'integrate out' the $\mathbf{x}$:

$$\text{Take residues} + Z^O \text{ rational} \implies \text{"Constant Term"}$$

- Try

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t = 0) = \mathrm{CT}\left[\mathbf{x}^{\mathbf{y}'} Z^O(\mathbf{x}; \mathbf{y}, 0)\right]$$

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$
- Need to:
  - 'Get rid' of $\mathbf{x}$
  - Add back initial heights $\mathbf{y}'$
- Idea: 'integrate out' the $\mathbf{x}$:

$$\text{Take residues} + Z^O \text{ rational} \implies \text{“Constant Term”}$$

- Try

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t = 0) = \mathrm{CT}\left[\mathbf{x}^{\mathbf{y}'}\, Z^O(\mathbf{x}; \mathbf{y}, 0)\right]$$

- But its no good.

Initial Condition

- Initial condition: $Z(\mathbf{y}, \mathbf{y}'; t = 0) = \prod_{i=1}^{N} \delta_{y_i, y_i'}$
- Need to:
  - 'Get rid' of $\mathbf{x}$
  - Add back initial heights $\mathbf{y}'$
- Idea: 'integrate out' the $\mathbf{x}$:

$$\text{Take residues} + Z^O \text{ rational } \implies \text{“Constant Term”}$$

- Try

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t = 0) = \mathrm{CT}\left[\mathbf{x}^{\mathbf{y}'} Z^O(\mathbf{x}; \mathbf{y}, 0)\right]$$

- But its no good.
- Need more solutions...

- Z^O with $x_i \to \frac{1}{x_i}$ still a solution

- Z^O with $x_i \rightarrow \dfrac{1}{x_i}$ still a solution
- Gives 2^N more solutions

- Z^O with $x_i \to \frac{1}{x_i}$ still a solution
- Gives 2^N more solutions
- Interesting: $2^N N! \implies$ Weyl group of signed permutations.

- Z^O with $x_i \to \frac{1}{x_i}$ still a solution
- Gives 2^N more solutions
- Interesting: $2^N N! \implies$ Weyl group of signed permutations.
- Linear combination

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t = 0) = \mathrm{CT}\left[\sum_{\chi} c_{\chi} \mathbf{x}^{\chi \cdot \mathbf{y}'} \, Z^O\left(x_1^{\chi_1}, x_2^{\chi_2}, \ldots, x_N^{\chi_N}; \mathbf{y}, 0\right)\right]$$

where $\chi_i = \pm 1$ and $\mathbf{x}^{\chi \cdot \mathbf{y}'} = \prod_i x_i^{\chi_i y_i'}$.

- Z^O with $x_i \to \frac{1}{x_i}$ still a solution
- Gives 2^N more solutions
- Interesting: $2^N N! \implies$ Weyl group of signed permutations.
- Linear combination

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t = 0) = \mathrm{CT}\left[\sum_{\chi} c_{\chi} \mathbf{x}^{\chi \cdot \mathbf{y}'} \; Z^O\left(x_1^{\chi_1}, x_2^{\chi_2}, \ldots, x_N^{\chi_N}; \mathbf{y}, 0\right)\right]$$

where $\chi_i = \pm 1$ and $\mathbf{x}^{\chi \cdot \mathbf{y}'} = \prod_i x_i^{\chi_i y_i'}$.

- Look for patterns for c_{χ}'s (then prove by induction)

- Z^O with $x_i \to \dfrac{1}{x_i}$ still a solution
- Gives 2^N more solutions
- Interesting: $2^N N! \implies$ Weyl group of signed permutations.
- Linear combination

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t=0) = \mathrm{CT}\left[\sum_{\chi} c_\chi \mathbf{x}^{\chi\cdot\mathbf{y}'}\, Z^O\left(x_1^{\chi_1}, x_2^{\chi_2}, \ldots, x_N^{\chi_N}; \mathbf{y}, 0\right)\right]$$

where $\chi_i = \pm 1$ and $\mathbf{x}^{\chi\cdot\mathbf{y}'} = \prod_i x_i^{\chi_i y_i'}$.

- Look for patterns for c_χ's (then prove by induction)
- To give...

Theorem

The total number of t-step 'watermelon' osculating paths starting at $\mathbf{y}'$ and ending at $\mathbf{y}$ is given by

$$Z(\mathbf{y} \leftarrow \mathbf{y}'; t, \omega) = CT\left[\Lambda_n^t \sum_{\chi} c_\chi \sum_{\sigma \in S_n} A_\sigma(\mathbf{x}, \omega) \prod_{i=1}^{n} x_i^{\chi_i (y_{\bar{\sigma}(i)} - y_i')}\right]$$

where $\chi_i = \pm 1$, and

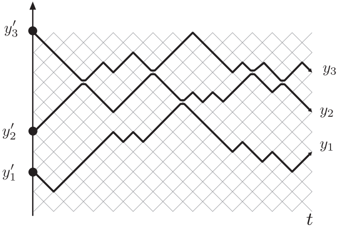
$$c_\chi = \begin{cases} 1 & \textit{if } \chi = (-1, \ldots, -1, \chi_i, -1, \ldots, -1) : \chi_i = +1,\ 1 \leqslant i \leqslant \frac{n+1}{2} \\ -1 & \textit{if } \chi = (-1, \ldots, -1, \chi_i, -1, \ldots, -1) : \chi_i = +1,\ \frac{n+1}{2} < i \leqslant n \\ 0 & \textit{otherwise} \end{cases}$$

for n odd (and similar for even) and $\Lambda_n = \prod_{i=1}^{n}(x_i + x_i^{-1})$.

For proof see: Brak & Wellington arXiv:1207.5268

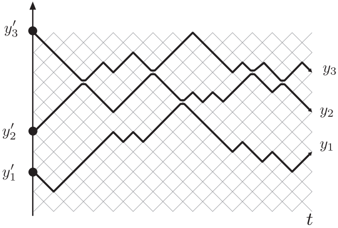
Non-intersecting

- Corollary: $\omega = 0 \implies$ non-intersecting



Non-intersecting

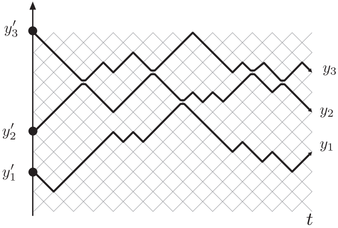
- Corollary: $\omega = 0 \implies$ non-intersecting



- $A_\sigma(\omega = 0) = \text{sign}(\sigma)$

Non-intersecting

- Corollary: $\omega = 0 \implies$ non-intersecting



- $A_\sigma(\omega = 0) = \mathsf{sign}(\sigma)$
- $Z(\mathbf{y} \leftarrow \mathbf{y}'; t)$ is a determinant.

– Thank You–