On the K-theory classification of topological states of matter

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Topological phases

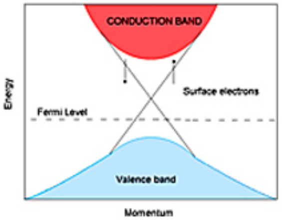
In contrast to usual phases, which are related to a spontaneously broken symmetry, topological phases (e.g. topological insulators) are many fermion systems possessing an unusual band structure that leads to a bulk band gap as well as topologically protected gapless extended surface modes.

Topological phases of free fermion models arise from symmetries of one-particle Hamiltonians (time reversal, particle-hole). There are 10 symmetry classes of Hamiltonians (the ‘ten-fold way’) and non trivial topological phases are classified by K-theory.

References

- A. Kitaev, arXiv:0901.2686
- M. Stone, C.-K. Chiu and A. Roy, arXiv:1005.3213
- D. Freed and G. Moore, arXiv:1208.5055
- M.Z. Hasan and C.L. Kane, arXiv:1002.3895

Topological phases, cont'd



In the presence of translation symmetry, we can block diagonalise the Hamiltonian in terms of eigenvalues under the translation operators

$$H = \bigoplus_{\mathbf{k} \in \mathrm{BZ}} H(\mathbf{k})$$

where $H(\mathbf{k})$ is so-called Bloch Hamiltonian, and BZ is the Brillouin zone (e.g. a torus $\mathbb{T}^d$).

Bands can have nontrivial structure protected under (gap-preserving) deformations of Hamiltonians. I.e. we need to classify deformation classes of Hamiltonians.
It suffices to put the gap at $E = E_F = 0$ and to study ‘flattened Hamiltonians’, i.e. with eigenvalues ± 1.

Flattened Hamiltonians

If we have an arbitrary gapped Hamiltonian H (with a gap at 0), let $P_\pm$ be the projection operator on the positive/negative eigenspace. The flattened Hamiltonian $\tilde{H}$, with eigenvalues ± 1, is defined as

$$\tilde{H} = P_+ - P_- = 1 - 2P_- \,.$$

To show that H and $\tilde{H}$ are homotopic, let P_λ be the projection operator onto the eigenspace of eigenvalue λ. We have

$$P_+ = \bigoplus_{\lambda>0} P_\lambda \,, \qquad P_- = \bigoplus_{\lambda<0} P_\lambda$$

Now consider

$$H_t = \bigoplus_\lambda \left(\frac{\lambda}{(1-t) + t|\lambda|} \right) P_\lambda \,, \qquad t \in [0,1] \,.$$

Then

$$H_0 = \bigoplus_\lambda \lambda P_\lambda = H \,, \qquad H_1 = \bigoplus_\lambda \frac{\lambda}{|\lambda|} P_\lambda = \bigoplus_{\lambda>0} P_\lambda - \bigoplus_{\lambda<0} P_\lambda = \tilde{H} \,.$$

The Example

Consider the Hamiltonian

$$H = \widehat{\mathbf{x}} \cdot \sigma = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

acting on $\mathbb{C}^2$, with $\widehat{\mathbf{x}} = (x, y, z) \in S^2$. We have

$$\mathrm{Tr}\, H = 0\,, \qquad H^\dagger = H\,, \qquad H^2 = 1\,,$$

from which we conclude that H has eigenvalues ± 1, each with multiplicity 1. For eigenvalue $\lambda = -1$, the normalised eigenvectors $\psi_-^{N/S}$ on $S^2_{N/S}$, where $S^2_N = S^2 \backslash \{z = -1\}$ and $S^2_S = S^2 \backslash \{z = 1\}$, are given by

$$\psi_-^N = \frac{1}{\sqrt{2(1+z)}} \begin{pmatrix} x - iy \\ -(1+z) \end{pmatrix}\,, \qquad \psi_-^S = \frac{1}{\sqrt{2(1-z)}} \begin{pmatrix} -(1-z) \\ x + iy \end{pmatrix}$$

Together they define a linebundle E_- over S^2, with first Chern class $c_1 = 1$. [Associated circle bundle is the Hopf fibration.]

The Example, cont'd

Knowing the eigenbundle E_-, we can reconstruct the Hamiltonian as follows. First we determine the projection operator $P_- : E \to E_-$, where E is the trivial $\mathbb{C}^2$-bundle over S^2

$$P_- = \psi_-^N \psi_-^{N\dagger} = \frac{1}{2} \begin{pmatrix} 1-z & -(x-iy) \\ -(x+iy) & 1+z \end{pmatrix} ,$$

and hence

$$H = P_+ - P_- = 1 - 2P_-$$

The Example, cont'd

A connection A_- on E_- is given, locally on $S^2_{N/S}$, by

$$A^N_- = i\psi^{N\dagger}_- d\psi^N_- = \frac{xdy - ydx}{2(1+z)} = \frac{\sin^2\theta\, d\phi}{2(1+\cos\theta)} = \tfrac{1}{2}(1-\cos\theta)\, d\phi$$

$$A^S_- = i\psi^{S\dagger}_- d\psi^S_- = \frac{-xdy + ydx}{2(1-z)} = \frac{-\sin^2\theta\, d\phi}{2(1-\cos\theta)} = -\tfrac{1}{2}(1+\cos\theta)\, d\phi$$

which is precisely the connection for a Dirac monopole.
On $S^2_N \cap S^2_S$ the $A^{N/S}_-$ differ by a gauge transformation

$$A^N_- - A^S_- = d\phi\,.$$

Thus

$$F_- = dA^N_- = dA^S_- = \tfrac{1}{2}\sin\theta\, d\theta \wedge d\phi\,,$$

is globally defined on S^2, and

$$c_1 = \frac{1}{2\pi}\int_{S^2} F_- = \frac{1}{4\pi}\mathrm{Vol}(S^2) = 1\,.$$

Projection operators and Berry connections

Let Ψ be an $N \times k$ matrix of k (orthonormal) vectors in $\mathbb{C}^N$. In terms of matrix components Ψ_{Aa}, $A = 1, \ldots, N$, $a = 1, \ldots, k$. We have

$$\Psi^\dagger \Psi = 1 \,.$$

The projections operator P onto the subspace spanned by the vectors Ψ_a, is given by

$$P = \Psi\Psi^\dagger \,, \qquad P^2 = P \,.$$

Now consider the subbundle $E \subset X \times \mathbb{C}^N$ given by P.
On E, we can canonically construct two connections ∇

- $\nabla s = Pd(Ps)$, with curvature $F_\nabla = P\, dP \wedge dP$.
- $Ds = \Psi^\dagger d(\Psi s) = ds + (\Psi^\dagger d\Psi)s$, with curvature $F_D = d\Psi^\dagger \wedge d\Psi + \Psi^\dagger d\Psi \wedge \Psi^\dagger d\Psi$ (Berry connection).

Projection operators and Berry connections, cont'd

They are related by

$$F_{\nabla} = \Psi F_D \Psi^{\dagger}$$

In particular we find

$$\mathrm{Tr}({F_{\nabla}}^n) = \mathrm{Tr}(P(dP)^{2n}) = \mathrm{tr}({F_D}^n)\,.$$

where Tr is taken over $\mathbb{C}^N$ and tr over $\mathbb{C}^k$.

Note: Of course, once we have a projection we should be able to associate that with a class in K-theory directly!

The Example, cont'd

In particular, for $P = \frac{1}{2}(1 - H)$,

$$c_1 = \frac{1}{2\pi} \int \mathrm{Tr}(P\, dP \wedge dP) = -\frac{1}{16\pi} \int \mathrm{Tr}(H\, dH \wedge dH)$$

E.g.

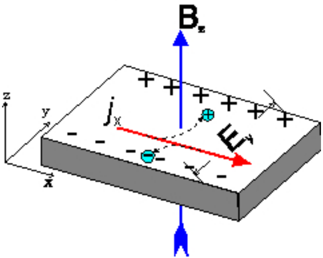
$$H = \widehat{h}(\mathbf{x}) \cdot \sigma \,, \qquad \widehat{h} : X \to S^2$$

gives negative eigenvector bundle with

$$c_1 = \frac{1}{8\pi} \int_{S^2} d^2x \; \epsilon^{\mu\nu} \widehat{h} \cdot (\partial_\mu \widehat{h} \times \partial_\nu \widehat{h})$$

[winding number of $\widehat{h}$, e.g. element of $\pi_2(S^2) \cong \mathbb{Z}$ for $X = S^2$.]

Integer Quantum Hall Effect



The Kubo formula for the Hall conductance σ_{xy}

$$j_x = \sigma_{xy} E_y$$

gives

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} n$$

where

$$n = c_1 = \frac{1}{2\pi} \int_{BZ} \operatorname{tr} F_D$$

Integer Quantum Hall Effect, cont'd

Determine deformation classes of Hamiltonians only up to addition of trivial valence bands (physical properties are the same). I.e. to the negative eigenbundle E_- we associate its class in $K^0(X)$.

Classifying spaces

We may parametrize our Hamiltonian as

$$H = A(\mathbf{x})\sigma_z A(\mathbf{x})^\dagger$$

where $A : X \to U(2)$. In fact, since $U(1) \times U(1) \subset U(2)$ commutes with σ_z, we have

$$A : X \to U(2)/U(1) \times U(1) \cong S^2 \,.$$

Note that, for $N \to \infty$, the symmetric space $\oplus_k U(N)/U(k) \times U(N-k)$ approaches the classifying space C_0,

$$\mathrm{K}^0(X) = [X, C_0]$$

In particular $[pt, C_0] \cong \pi_0(C_0) \cong \mathbb{Z}$, $[S^2, C_0] \cong \pi_2(C_0) \cong \mathbb{Z}$.

Symmetries

- Time Reversal Symmetry (TRS):
  $TH(\mathbf{k})T^{-1} = H(-\mathbf{k})$, $T^2 = \pm 1$ (anti-linear)
- Particle-Hole Symmetry (PHS) (Charge Conjugation):
  $PH(\mathbf{k})P^{-1} = -H(-\mathbf{k})$, $P^2 = \pm 1$ (anti-linear)
- Sublattice Symmetry (SLS) (Chiral):
  $CH(\mathbf{k}) = -H(\mathbf{k})C$, $C = PT$

There are 3×3 possible choices for T^2, P^2, denoted as $0, \pm 1$, and for $T = P = 0$, there are two choices for C, denoted as $0, 1$.

This leads to 10 symmetry classes [Dyson, Altand-Zirnbauer]

Altland-Zirnbauer classes and The Periodic Table

AZ label	TRS	PHS	SLS	$d=0$	$d=1$	$d=2$	$d=3$
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	$+1$	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	$+1$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	0	$\mathbb{Z}$	0
CI	$+1$	-1	1	0	0	0	$\mathbb{Z}$
AI	$+1$	0	0	$\mathbb{Z}$	0	0	0

Classifying spaces

AZ label	Class. Space	G/H	π_0
A	C_0	$\bigoplus_k (U(N)/U(N-k) \times U(k))$	$\mathbb{Z}$
AIII	C_1	$U(N) \times U(N)/U(N)$	0
BDI	R_0	$\bigoplus_k (O(N)/O(N-k) \times O(k)$	$\mathbb{Z}$
D	R_1	$O(N) \times O(N)/O(N)$	$\mathbb{Z}_2$
DIII	R_2	$O(2N)/U(N)$	$\mathbb{Z}_2$
AII	R_3	$U(2N)/Sp(N)$	0
CII	R_4	$\bigoplus_k (Sp(N)/Sp(N-k) \times Sp(k))$	$\mathbb{Z}$
C	R_5	$Sp(N) \times Sp(N)/Sp(N)$	0
CI	R_6	$Sp(N)/U(N)$	0
AI	R_7	$U(N)/O(N)$	0

Real K-theory

The R_q are the classifying spaces for Atiyah's real K-theory

$$\mathrm{KR}^{-q}(X) = [X, R_q]$$

Examples

$$\mathrm{KR}^{-q}(S^d) \cong \pi_0(R_{q-d})$$

and

$$\mathrm{KR}^{-q}(\mathbb{T}^d) \cong \bigoplus_{n=0}^{d} \binom{d}{n} \pi_0(R_{q-n})$$

Clifford algebras

$\mathrm{Cl}^{p,q}$ is the algebra (over $\mathbb{R}$) generated by $e_i, i = 1, \ldots, p+q$, with

$$\begin{aligned} e_i^2 &= -1 & i &= 1, \ldots, p \\ e_i^2 &= 1 & i &= p+1, \ldots, p+q \\ e_i e_j + e_j e_i &= 0 & i &\neq j \end{aligned}$$

We have the folllowing isomorphisms

$$\begin{aligned} \mathrm{Cl}^{p,0} \otimes \mathrm{Cl}^{0,2} &\cong \mathrm{Cl}^{0,p+2} \\ \mathrm{Cl}^{0,p} \otimes \mathrm{Cl}^{2,0} &\cong \mathrm{Cl}^{p+2,0} \\ \mathrm{Cl}^{p,q} \otimes \mathrm{Cl}^{1,1} &\cong \mathrm{Cl}^{p+1,q+1} \\ \mathrm{Cl}^{p+8,0} &\cong \mathrm{Cl}^{p,0} \otimes \mathbb{R}[16] \end{aligned}$$

For Clifford algebras over $\mathbb{C}$ we have $\mathrm{Cl}^{p+2} \cong \mathrm{Cl}^{p} \otimes \mathbb{C}[2]$.

Extension of Clifford Modules

Suppose we have a representation of $\mathrm{Cl}^{k,0}$ in $O(16r)$

$$J_iJ_j + J_jJ_i = -2\delta_{ij}$$

Let G_1 be the subgroup of $O(16r)$ that commutes with J_1, G_2 the subgroup of G_1 that commutes with J_2, etc. We get the following chain of subgroups

$$\begin{aligned} O(16r) \underset{R_2}{\supset} U(8r) \underset{R_3}{\supset} Sp(4r) \underset{R_4}{\supset} Sp(2r) \times Sp(2r) \underset{R_5}{\supset} Sp(2r) \\ \underset{R_6}{\supset} U(2r) \underset{R_7}{\supset} O(2r) \underset{R_0}{\supset} O(r) \times O(r) \underset{R_1}{\supset} O(r) \supset \ldots \end{aligned}$$

Subsequent quotients parametrize the extensions of $\mathrm{Cl}^{p,0}$ to $\mathrm{Cl}^{p+1,0}$. These are precisely the symmetric spaces (classifying spaces) encountered before.
Similarly in the complex case

$$\ldots \supset U(2r) \underset{C_0}{\supset} U(r) \times U(r) \underset{C_1}{\supset} U(r) \supset \ldots$$

THANKS