A variational approach for exactly solvable BEC-BCS crossover Hamiltonians

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Outline

- Introduce a general class of BEC-BCS crossover Hamiltonians
  - What constraints on the coupling parameters are sufficient to ensure they are exactly solvable?
- The standard method of solution is to construct integrable Hamiltonians via the Quantum Inverse Scattering Method (QISM)
  - This is an excellent way to find exactly solvable models but is not **always** well suited to determine the most general ones.
- We have taken a direct approach which enables general classes of exactly solvable Hamiltonians to be determined.
  - We have found three previously unknown exactly solvable models and been able to classify seven known integrable models within two general classes.

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Variational BEC-BCS crossover Hamiltonians

For real-valued $f(z_k)$, and complex-valued $g(z_k)$,

$$H = \alpha N_0 + \sum_{k=1}^{L} f(z_k) N_k + \kappa N_0^2 - \beta \sum_{k=1}^{L} g(z_k) b_0 b_k^\dagger - \beta \sum_{k=1}^{L} \overline{g(z_k)} b_0^\dagger b_k - \sigma \sum_{k,s}^{L} g(z_k) \overline{g(z_s)} b_k^\dagger b_s$$

where $N_i = b_i^\dagger b_i$ and $b_k^\dagger$, $k = 1, 2, \dots L$ are the hard-core Cooper pair creation operators, $b_0^\dagger$ is the bosonic creation operator:

$$b_k^\dagger = c_k^\dagger c_{-k}^\dagger, \quad [b_k, b_k^\dagger] = I - 2N_k, \quad (k = 1, 2, \dots L)$$
$$[b_0, b_0^\dagger] = I, \quad [b_i, b_j^\dagger] = 0 \quad (i \neq j).$$
$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij}, \quad \forall i, j = \pm k$$

For the standard fermion operators $c_{\pm k}^\dagger$ and $c_{\pm k}$, $k = 1, 2, \dots L$.

We assume there are no unpaired fermions. ▸ Case 1 ▸ Case 2

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Quantum Inverse Scattering Method

Through QISM we construct a parameter dependant operator $t(u)$ acting on a vector space W representing the Hilbert space of physical states such that

$$[t(u), t(v)] = 0, \quad \forall\ u, v.$$

We refer to this parameter as the spectral parameter. This enables a series expansion in the spectral parameter

$$t(u) = \sum_k t^{(k)} u^k$$

$$\left[t^{(j)}, t^{(k)}\right] = 0, \quad \forall\ j, k.$$

Each $t^{(k)}$ represents a constant of the motion for any Hamiltonian expressible as a function of the operators $t^{(k)}$.

Integrability

The Hamiltonian is integrable as long as the number of degrees of freedom is equal to the number of conserved quantities.

Constructing the transfer matrices

To construct the transfer matrices we take a solution of the Yang-Baxter equation (YBE):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \;\; \forall\, u, v,$$

for $R \in \mathrm{End}(V \otimes V)$ where $R_{12} = R \otimes \mathrm{id}$. From the R-matrix we define the Yang-Baxter algebra (YBA) which is generated by elements of the $n \times n$ monodromy matrix $T(u)$ which satisfies:

$$R_{12}(u)T_{13}(u+v)T_{23}(v) = T_{23}(v)T_{13}(u+v)R_{12}(u) \;\; \forall\, u, v.$$

The Yang-Baxter algebra has a bialgebra structure enabling us to construct monodromy matrices acting on $V^{\otimes n}$.

Transfer Matrices

The commuting family of transfer matrices is given by the matrix trace $t(u) = \pi\,[\mathrm{tr}\, T(u)]$, where π is a realization of the YBA.

Or, we can solve H directly!

To what extent can an exact solution be found for H?

$$H = \alpha N_0 + \sum_{k=1}^{L} f(z_k) N_k + \kappa N_0^2 - \beta \sum_{k=1}^{L} g(z_k) b_0 b_k^\dagger$$
$$-\beta \sum_{k=1}^{L} \overline{g(z_k)} b_0^\dagger b_k - \sigma \sum_{k,s}^{L} g(z_k) \overline{g(z_s)} b_k^\dagger b_s$$

We assume the eigenstates of H are of the form

$$|\Psi\rangle = \prod_{j=1}^{M} C(y_j)|0\rangle, \quad C(y) = \gamma(y) b_0^\dagger + \sum_{k=1}^{L} h(y, z_k) b_k^\dagger$$

and $h(y,z)$ is yet to be determined. The goal is to determine the action of the Hamiltonian on $|\Psi\rangle$ and choose constraints in the coupling parameters and $h(y,z)$ such that

$$H|\Psi\rangle = \lambda|\Psi\rangle.$$

The following commutation relations can be determined

$$[b_0, C(y)] = \gamma(y)I, \quad [N_0, C(y)] = \gamma(y)b_0^\dagger, \left[N_0^2, C(y)\right] = \gamma(y)b_0^\dagger(I + 2N_0),$$

$$[b_k, C(y)] = h(y, z_k)(I - 2N_k), \quad [N_k, C(y)] = h(y, z_k)b_k^\dagger.$$

The action of H on $|\psi\rangle$ can then be calculated ▸ $H|\psi\rangle$. In general there will be terms that are linearly independant of the "eigenstate"

$$H|\psi\rangle = \lambda|\psi\rangle + |\phi\rangle, \quad \langle\phi|\psi\rangle = 0.$$

Using some algebra and choosing appropriate constraints on the system we can isolate these terms. The compatability of these constraints along with a condition that these "unwanted terms" cancel are the solvability conditions of the Hamiltonian.

We found the following constraints provided solvability:

$$\overline{g(z_s)}h(y_j,z_s)h(y_l,z_s) = k(y_j,y_l)h(y_l,z_s) + k(y_l,y_j)h(y_j,z_s), \tag{1}$$

$$(y_j-\alpha-\kappa)\gamma(y_j) + \beta\sum_{k=1}^{L}\overline{g(z_k)}h(y_j,z_k) = 2\beta\sum_{l\neq j}^{M}k(y_j,y_l) \tag{2}$$

$$\beta\gamma(y_j) - r(y_j) + \sigma\sum_{k=1}^{L}\overline{g(z_k)}h(y_j,z_k) = 2\sigma\sum_{l\neq j}^{M}k(y_j,y_l) \tag{3}$$

$$\beta\left(k(y_j,y_l)\gamma(y_l) + k(y_l,y_j)\gamma(y_j)\right) = \kappa\gamma(y_j)\gamma(y_l) \tag{4}$$

$$\sigma\left(k(y_j,y_l)\gamma(y_l) + k(y_l,y_j)\gamma(y_j)\right) = 0. \tag{5}$$

For $\kappa \neq 0 \neq \sigma$ constraints (4) and (5) are incompatible and we have at least two separate cases.

Each case will have separate solvability conditions.

In the case $\sigma = 0 = \kappa$ where both of the above cases describe the same system, the solvability constraints are identical.

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Results: What constraints ensure solvability?

$$H = \alpha N_0 + \sum_{k=1}^{L} f(z_k) N_k + \kappa N_0^2 - \beta b_0 Q^\dagger - \beta b_0^\dagger Q - \sigma Q^\dagger Q, \quad Q^\dagger = \sum_{k=1}^{L} g(z_k) b_k^\dagger$$

Case 1: No Cooper Pair-Cooper Pair Interaction ($\sigma = 0$) ▸ H

- The manifold of exact solvability is

$$f(z_k) = \kappa^{-1}\beta^2 g(z_k)\overline{g(z_k)} + \kappa^{-1} c_1$$

where c_1 is a constant.

- The eigenstates are

$$|\Psi\rangle = \prod_{j=1}^{M}\left[\gamma(y_j) b_0^\dagger + \sum_{k=1}^{L} h(y_j, z_k) b_k^\dagger\right]|0\rangle, \quad h(y_j, z_k) = \frac{\beta\gamma(y_j) g(z_k)}{f(z_k) - y_j}.$$

and the Bethe ansatz equations are

$$y_j - (\alpha + \kappa) + \sum_{k=1}^{L} \frac{\kappa f(z_k) - c_1}{f(z_k) - y_j} = 2\sum_{l \neq j}^{M} \frac{c_1 - \kappa y_l}{y_j - y_l}.$$

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Case 2: No Quartic Term ($\kappa = 0$) H

- The manifold of exact solvability is

$$|g(z_k)|^2 + c_1 = c_2 f(z_k), \quad c_2(\sigma\alpha + \beta^2) = c_1 \sigma$$

  where c_2 is a constant.

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$$\frac{c_1(y_j - \alpha)}{c_2 - c_1 y_j} + \sum_{k=1}^{L} \frac{c_2 f(z_k) - c_1}{f(z_k) - y_j} = 2 \sum_{l \neq j}^{M} \frac{c_1 - c_2 y_l}{y_j - y_l}.$$

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Limiting Cases

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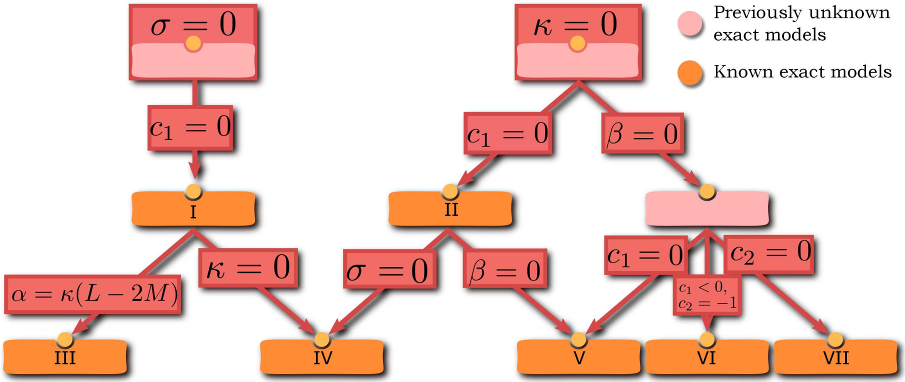

Limiting Cases

- In the case $\sigma = 0 = \kappa$ where both the above cases reduce to the same system, the have the same solvability constraints
- Integrability in seven sub-cases has been understood through QISM
  - I $p + ip$-wave BCS coupled to a bosonic mode with no pair-pair interactions (H. S. Lerma et al., Phys. Rev. B 84, 2011).
  - II $p + ip$-wave BCS coupled to a bosonic mode with no bosonic self-interaction (C. Dunning et al., Nucl. Phys. B 848, 2011).
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  - VII reduced s-wave BCS (R. W. Richardson, Phys. Lett. 3, 1963).
- Through a variational approach we have found three more general models of which the known models can be obtained by taking appropriate limits (with Jon Links and Phil Isaac, Inverse Problems 28, 2012).

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Limiting Cases

- In the case $\sigma = 0 = \kappa$ where both the above cases reduce to the same system, the have the same solvability constraints
- Integrability in seven sub-cases has been understood through QISM
  - I $p + ip$-wave BCS coupled to a bosonic mode with no pair-pair interactions (H. S. Lerma et al., Phys. Rev. B 84, 2011).
  - II $p + ip$-wave BCS coupled to a bosonic mode with no bosonic self-interaction (C. Dunning et al., Nucl. Phys. B 848, 2011).
  - III $p + ip$-wave BCS coupled to a bosonic mode, constructed from a boundary QISM (C. Dunning et al., Nucl. Phys B 748, 2006).
  - IV a Dicke-type model (M. Gaudin, J. Physique 37, 1976).
  - V the $p + ip$-wave BCS (Ibanez et al. Phys Rev B 79, 2009).
  - VI BCS for heavy nuclei (J. Dukelsky et al., Phys. Rev. C 84, 2011).
  - VII reduced s-wave BCS (R. W. Richardson, Phys. Lett. 3, 1963).
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Summary

- In the work presented above, we have assumed a separable ansatz for the general family of BEC-BCS crossover Hamiltonians and successfully determined sub-families of solvable Hamiltonians by imposing constraints on the coupling parameters of the system.
- We found two distinct families of solvable systems. The first is the family of pairing Hamiltonians with no coupling of Cooper-pair states ($\sigma = 0$). The second is the family of Hamiltonians with no quartic term ($\kappa = 0$). Both cases had to be treated separately since some of the constraints were incompatible in the general case.
- We were able to do this without resorting to any prior knowledge of a set of conserved operators or transfer matrix. This was made possible by formulating the eigenfunctions as factorisable operators acting on a suitable reference state, analogous to the algebraic Bethe ansatz, while taking a co-ordinate Bethe ansatz type approach to solve the Hamiltonian directly.

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Thanks for your attention!

return

$$
\begin{aligned}
H|\Psi\rangle = {} & (\alpha+\kappa)\sum_{j=1}^{M}\gamma(y_j)b_0^\dagger|\Psi_j\rangle - \beta\sum_{j=1}^{M}\sum_{k=1}^{L}\overline{g(z_k)}h(y_j,z_k)b_0^\dagger|\Psi_j\rangle \\
& +2\beta\sum_{j,l\neq j}^{M}k(y_j,y_l)b_0^\dagger|\Psi_j\rangle + \sum_{j=1}^{M}\sum_{k=1}^{L}f(z_k)h(y_j,z_k)b_k^\dagger|\Psi_j\rangle \\
& -\beta Q^\dagger\sum_{j=1}^{M}\gamma(y_j)|\Psi_j\rangle - \sigma Q^\dagger\sum_{j=1}^{M}\sum_{s=1}^{L}\overline{g(z_s)}h(y_j,z_s)|\Psi_j\rangle \\
& +2\sigma Q^\dagger\sum_{j,l\neq j}^{M}k(y_j,y_l)|\Psi_j\rangle - 2\sigma Q^\dagger\sum_{j,l\neq j}^{M}k(y_j,y_l)\gamma(y_l)b_0^\dagger|\Psi_{jl}\rangle \\
& -2\beta\sum_{j,l\neq j}^{M}k(y_j,y_l)\gamma(y_l)b_0^\dagger b_0^\dagger|\Psi_{jl}\rangle + \kappa\sum_{j=1,l\neq j}^{M}\gamma(y_j)\gamma(y_l)b_0^\dagger b_0^\dagger|\Psi_{jl}\rangle.
\end{aligned}
$$

where $Q^\dagger = \sum_{k=1}^{L} g(z_k)b_k^\dagger$ and

$$
|\Psi_j\rangle = \prod_{l\neq j}^{M} C(y_l)|0\rangle, \quad |\Psi_{ij}\rangle = \prod_{l\neq i,j}^{M} C(y_l)|0\rangle.
$$