Integrability as a consequence of Discrete Holomorphicity

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Outline

- Discrete holomorphicity
- Self-dual Potts model
- Yang-Baxter integrability
- Holomorphic observable
- **DH $\implies$ YBE**

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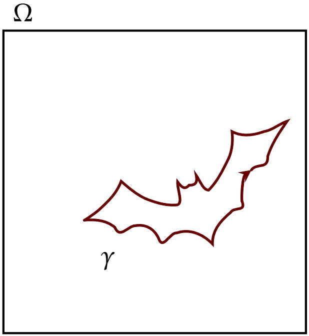
Discrete Holomorphicity

Discretized analyticity



$\psi : \Omega \to \mathbb{C}$

analytic

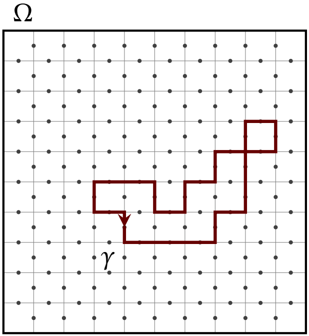
$\Downarrow$

$$\oint_\gamma \psi(z)\,\mathrm{d}z = 0$$

[1] Smirnov S *Ann. Math.* **172** 101 (2010)

Discrete Holomorphicity

Discretized analyticity



$\psi : \Omega \to \mathbb{C}$

analytic

$$\Downarrow$$

$$\oint_\gamma \psi(z)\,\mathrm{d}z = 0$$

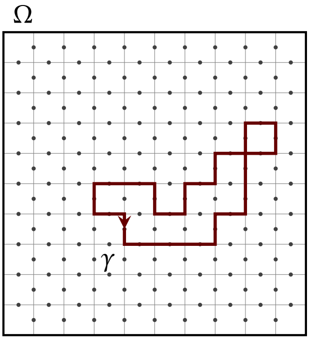
Discretize!

$$\sum_\gamma \psi(z)\,\Delta z \approx 0$$

[1] Smirnov S *Ann. Math.* **172** 101 (2010)

Discrete Holomorphicity

Discretized analyticity



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Discretize!

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discretely holomorphic[1]

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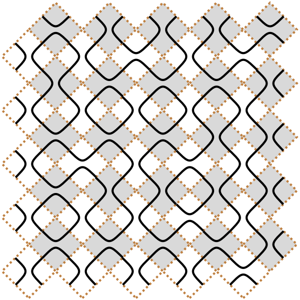
Self-dual Potts model

Loop formulation



two kinds of **tiles**

type a

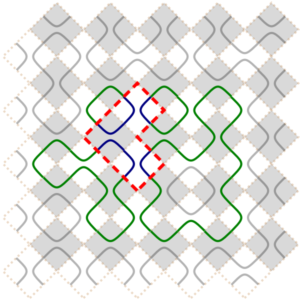
type b

statistical **weight**, $w(\gamma)$ is

$$n = \sqrt{Q} = 2\cos\lambda \qquad (0 \leq Q \leq 4)$$

Self-dual Potts model

Loop formulation



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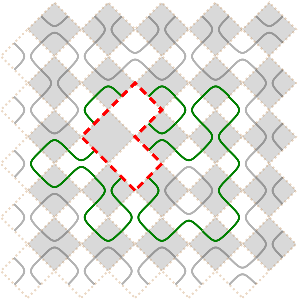

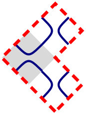

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statistical **weight**, $w(\gamma)$ is

$a^{\#}$() $b^{\#}$() $\tilde{a}^{\#}$() $\tilde{b}^{\#}$() $n^{\#(\bigcirc)}$

$$n = \sqrt{Q} = 2\cos\lambda \qquad (0 \leq Q \leq 4)$$

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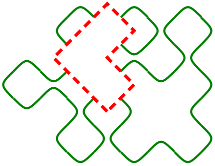
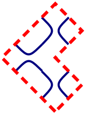
Self-dual Potts model

Loop formulation





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Self-dual Potts model

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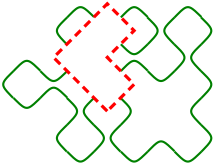

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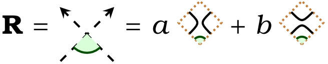
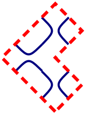
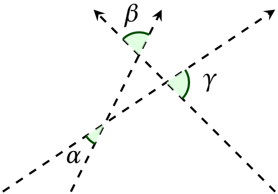




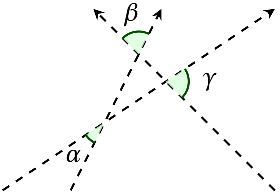
Integrability

Yang-Baxter equation



INTEGRABILITY

YANG-BAXTER EQUATION



$a_\alpha \, a_\beta \, a_\gamma$  $+ \, a_\alpha \, a_\beta \, b_\gamma$  $+ \, a_\alpha \, b_\beta \, a_\gamma$  $+ \, b_\alpha \, a_\beta \, a_\gamma$ 

$+ \left(a_\alpha \, b_\beta \, b_\gamma + b_\alpha \, a_\beta \, b_\gamma + b_\alpha \, b_\beta \, a_\gamma \right)$  $+ \, b_\alpha \, b_\beta \, b_\gamma$ 

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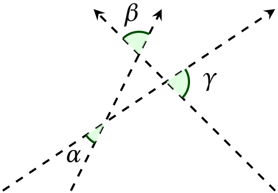
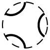
INTEGRABILITY

YANG-BAXTER EQUATION

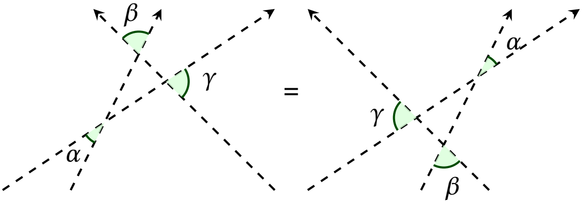


$a_\alpha\, a_\beta\, a_\gamma$  $+\, a_\alpha\, a_\beta\, b_\gamma$  $+\, a_\alpha\, b_\beta\, a_\gamma$  $+\, b_\alpha\, a_\beta\, a_\gamma$ 

$+\left(a_\alpha\, b_\beta\, b_\gamma + b_\alpha\, a_\beta\, b_\gamma + b_\alpha\, b_\beta\, a_\gamma + n\, b_\alpha\, b_\beta\, b_\gamma\right)$ 

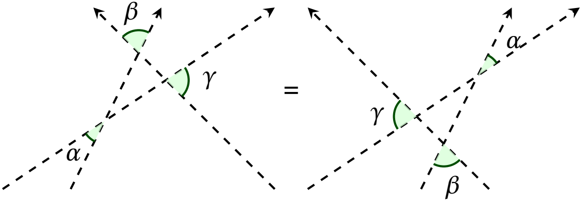
Integrability

Yang-Baxter equation



INTEGRABILITY

YANG-BAXTER EQUATION



$$\left(-a_\alpha\, a_\beta\, a_\gamma + a_\alpha\, b_\beta\, b_\gamma + b_\alpha\, a_\beta\, b_\gamma + b_\alpha\, b_\beta\, a_\gamma + n\, b_\alpha\, b_\beta\, b_\gamma\right)\Big($$  $-$  $\Big) = 0$

Yang-Baxter equation

$$a_\alpha\, a_\beta\, a_\gamma = a_\alpha\, b_\beta\, b_\gamma + b_\alpha\, a_\beta\, b_\gamma + b_\alpha\, b_\beta\, a_\gamma + n\, b_\alpha\, b_\beta\, b_\gamma$$

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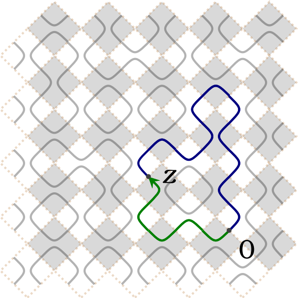
Holomorphic observable

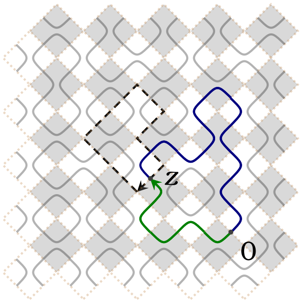


observable on **embedded** lattice[1]

$$\psi(z) = \sum_{\gamma \in \Gamma(0,z)} e^{-i\sigma \, W_\gamma(0 \to z)} \, w(\gamma)$$

[1] Riva V & Cardy J *J. Stat. Mech.* P12001 (2006)

HOLOMORPHIC OBSERVABLE



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contour sum **vanishes**

$$\sum_{\text{contour}} \psi(z)\, \Delta z = 0$$

at the **isotropic** point

$$a = b = \tilde{a} = \tilde{b}$$

with

$$\sigma = 1 - \frac{2\lambda}{\pi}$$

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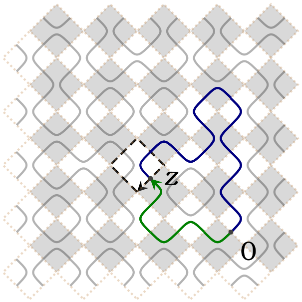
Holomorphic observable



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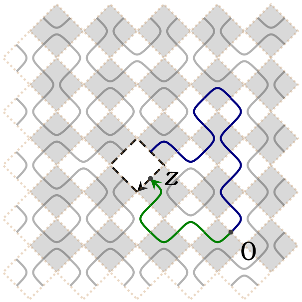

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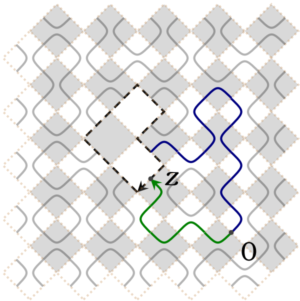

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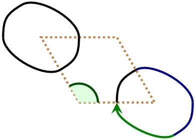
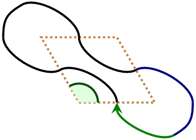
[1] Riva V & Cardy J *J. Stat. Mech.* P12001 (2006)

Discrete Holomorphicity

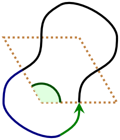
On a rhombus[1]

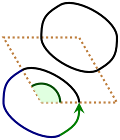
with $\phi(\alpha) = e^{i(1-\sigma)\alpha}$, **want** $\sum_{\diamond} \psi(z)\,\Delta z = 0$





$$n^2 \left(1 - \phi(\alpha - \pi)\right) a_\alpha \quad + \quad n \left(1 - \phi(\alpha - \pi) + \phi(-\pi) - \phi(\alpha)\right) b_\alpha = 0$$





$$n \left(1 - \phi(\alpha - \pi) + \phi(\pi) - \phi(\alpha)\right) a_\alpha \quad + \quad n^2 \left(1 - \phi(\alpha)\right) b_\alpha = 0$$

[1] Ikhlef Y & Cardy J *J. Phys. A* **42** 102001 (2009)

Discrete Holomorphicity

On a rhombus

- needed **angle independent** condition

$$\phi(\pi)+\phi(-\pi)=n^2-2$$

- **fixes** conformal spin

$$\cos((1-\sigma)\pi)=\cos(2\lambda)$$

$$\sigma=1-\frac{2\lambda}{\pi} \quad \mod 2\mathbb{Z}$$

- solution weights

$$\frac{a_\alpha}{b_\alpha}=\frac{\sin\left(\frac{\lambda}{\pi}\alpha\right)}{\sin\left(\frac{\lambda}{\pi}(\pi-\alpha)\right)}, \qquad \frac{a(u)}{b(u)}=\frac{\sin u}{\sin(\lambda-u)}$$

Discrete Holomorphicity

Self-duality

- **dual weights** given by $\alpha \mapsto \pi - \alpha$

$$\frac{\widetilde{a}_\alpha}{\widetilde{b}_\alpha} = \frac{a_{\pi-\alpha}}{b_{\pi-\alpha}} = \frac{\sin\left(\frac{\lambda}{\pi}(\pi-\alpha)\right)}{\sin\left(\frac{\lambda}{\pi}\alpha\right)}$$

- satisfies **self-duality**

$$\frac{a_\alpha}{b_\alpha}\,\frac{\widetilde{a}_\alpha}{\widetilde{b}_\alpha} = 1$$

- rôles of a and b are **interchanged**

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A family of solutions

- condition for discrete **holomorphicity**

$$\cos((1-\sigma)\pi) = \cos(2\lambda)$$

- **positive** solutions for conformal spin

$$\sigma = 1 + 2\left(\ell - \frac{\lambda}{\pi}\right)$$

- weights **indexed** by ℓ

$$\frac{a_\alpha}{b_\alpha} = (-1)^\ell \frac{\sin\left(\left(\frac{\lambda}{\pi} - \ell\right)\alpha\right)}{\sin\left(\left(\frac{\lambda}{\pi} - \ell\right)(\pi - \alpha)\right)}, \qquad \frac{a(u)}{b(u)} = (-1)^\ell \frac{\sin\left(\left(1 - \frac{\ell\pi}{\lambda}\right)u\right)}{\sin\left(\left(1 - \frac{\ell\pi}{\lambda}\right)(\lambda - u)\right)}$$

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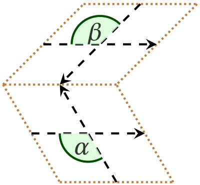
Inversion Relation

DH $\Longrightarrow$ IR



DH on **every** rhombus with **same** σ

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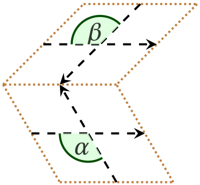
INVERSION RELATION

DH $\implies$ IR



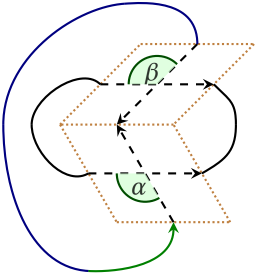
DH on **every** rhombus with **same** σ

internal edge **cancels**

five different chord diagrams

Inversion Relation

DH $\implies$ IR



DH on **every** rhombus with **same** σ

internal edge **cancels**

five different chord diagrams

one is enough

Inversion Relation

DH $\Longrightarrow$ IR

- **quadratic** equation in weights
- $\beta = -\alpha \quad \Longrightarrow$

$$n\, a_\alpha\, a_{-\alpha} + a_\alpha\, b_{-\alpha} + b_\alpha\, a_{-\alpha} = 0$$

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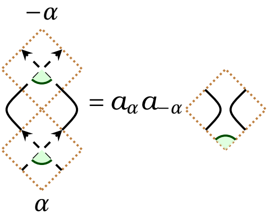
Inversion Relation

DH $\implies$ IR



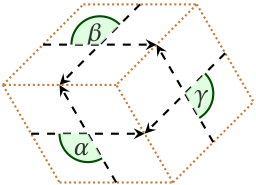
- **quadratic** equation in weights
- $\beta = -\alpha \quad \implies$

$$n\, a_\alpha\, a_{-\alpha} + a_\alpha\, b_{-\alpha} + b_\alpha\, a_{-\alpha} = 0$$

- the **inversion relation**
- $\beta = 2\pi - \alpha, \alpha \mapsto \pi - \alpha \quad \implies$ inversion relations for the **dual** weights

Yang-Baxter equations

DH $\implies$ YBE



- third rhombus with $\alpha + \beta + \gamma = 2\pi$
- internal edges **cancel**

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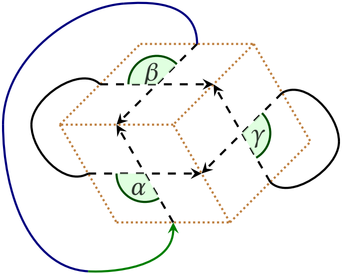
Yang-Baxter equations

DH $\implies$ YBE



- third rhombus with $\alpha + \beta + \gamma = 2\pi$
- internal edges **cancel**
- $a_\alpha\, a_\beta\, a_\gamma = a_\alpha\, b_\beta\, b_\gamma + b_\alpha\, a_\beta\, b_\gamma + b_\alpha\, b_\beta\, a_\gamma + n\, b_\alpha\, b_\beta\, b_\gamma$
- the **Yang-Baxter equation**!

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Summary

- **DH** $\Longrightarrow$ **IR** + **YBE**
- **DH** fixes **conformal spin**
- simple **geometric** construction

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THE END

Thank you![1]

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[1] Alam IT & Batchelor MT *J. Phys. A* **45** 494014 (2012)