A deterministic map for particle dynamics in polygonal billiards

Jordan Orchard[1]
Federico Frascoli[1]
Carlos Mejía-Monasterio[2]

[1]Swinburne University of Technology
[2]Technical University of Madrid

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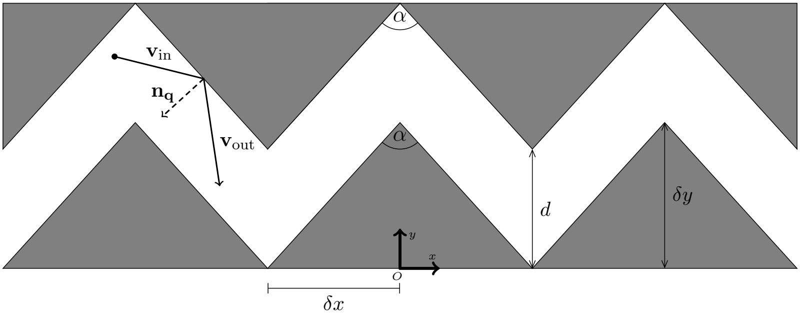
Parallel billiard channels

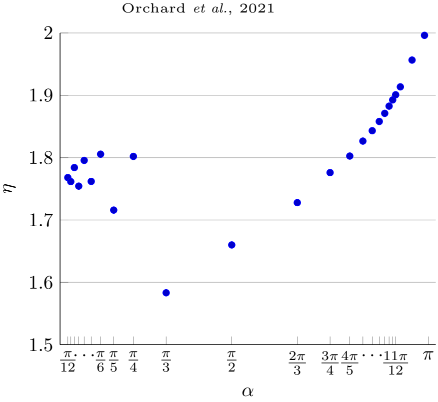
- Billiard family defined by parameters $\alpha \in (0, \pi]$ and $d \in (0, \infty)$
- Particles reflect specularly at the boundary:
  $\mathbf{v}_{\text{out}} = \mathbf{v}_{\text{in}} - 2(\mathbf{v}_{\text{in}} \cdot \mathbf{n}_{\mathbf{q}})\mathbf{n}_{\mathbf{q}}$



Geometry and transport

Initially localised 'droplet' of particles $\Delta x(t) = x(t) - x(0)$ spread out along the channel with density $P(\Delta x, t)$ and second moment $\langle \Delta x^2 \rangle \sim t^\eta$ characterising

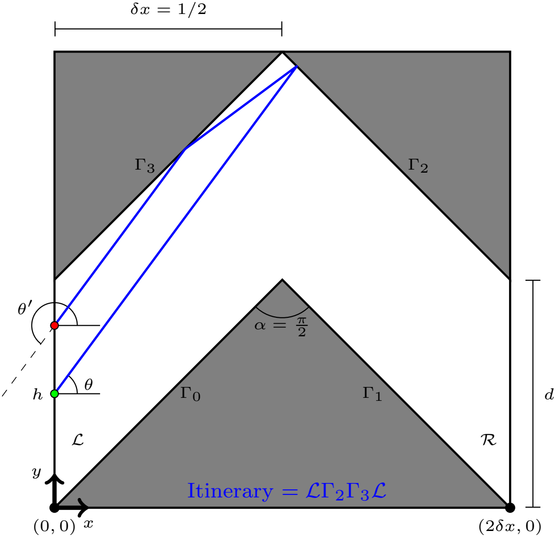
$$\begin{aligned} \text{subdiffusion when} \quad & 0 < \eta < 1 \\ \text{normal diffusion when} \quad & \eta = 1 \\ \text{superdiffusion when} \quad & 1 < \eta < 2 \\ \text{ballistic motion when} \quad & \eta = 2 \end{aligned}$$



Blue dots: $\alpha = \alpha_R = \frac{\pi}{12}, \frac{\pi}{11}, \ldots, \pi$

Elementary cell geometry

Fix $\alpha = \pi/2$ and $d = 1/2$:



A map encoding particle dynamics

- Let $\widehat{S} : \widehat{X} \to \widehat{X}$ be a map with $\widehat{X} = (0, d) \times \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

- By vertical symmetry of the elementary cell we have

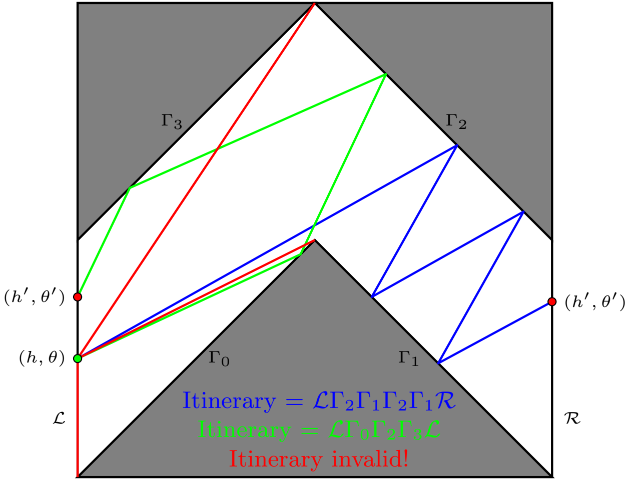
$$\widehat{S}(h, \theta) = \begin{cases} S(h, \theta) & \text{if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ S(h, \pi - \theta) & \text{if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

where $S : X \to \widehat{X}$ and $X = (0, d) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- How to define S?

Singular trajectories

Intuition: Between two singular trajectories exist a regular trajectory.



Singular trajectories

- The phase space is a finite union of subsets, each representing families of regular trajectories that have distinct itineraries:

$$X = \bigcup_{j=0}^{M_d-1} \widetilde{X}_j$$

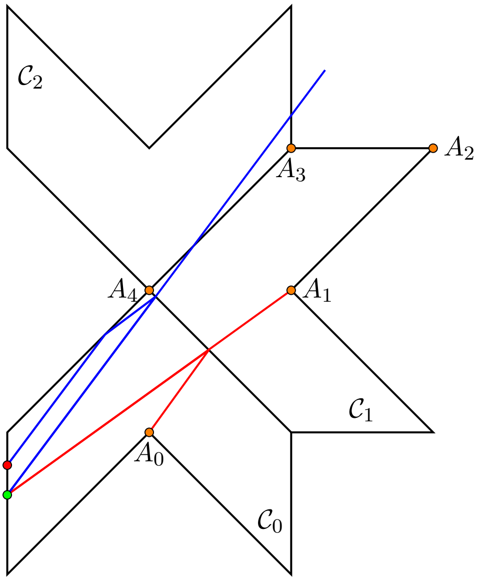
and $\mathcal{G} = \bigcup_{j=0}^{M_d-1} \partial\widetilde{X}_j$ is a set of zero area on X

- The $\partial\widetilde{X}_j$ are curves on X that parameterise singular trajectories!
- Thus, $\widetilde{X}_j = \mathcal{D}_j \times \Pi_j$ where $\mathcal{D}_j \subset (0, d)$,

$$\Pi_j = \{\theta : h \in \mathcal{D}_j \ \wedge \ \xi_j(h) \leq \theta \leq \xi_{j+1}(h)\}$$

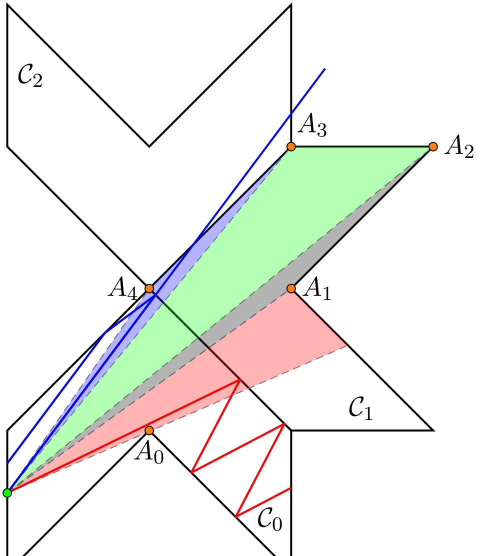
and $\xi_j : \mathcal{D}_j \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Detour: Unfolding

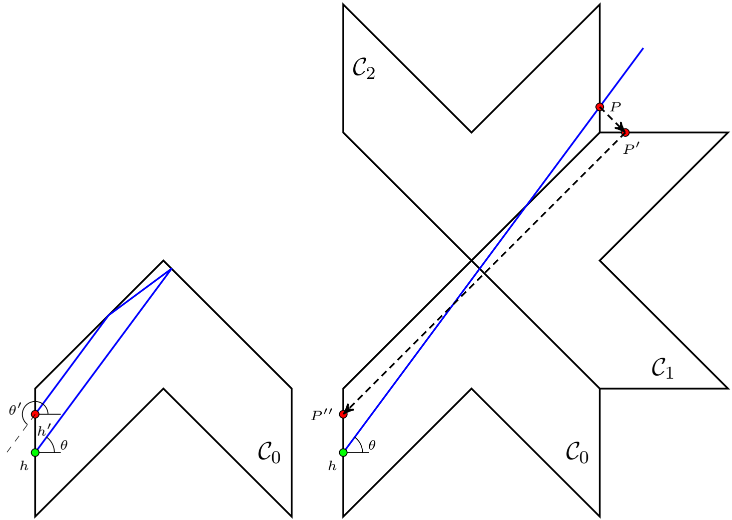


Detour: Unfolding

Itineraries encode the unfoldings: $\mathcal{C}_n = \mathrm{Refl}(\mathcal{C}_0, \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_n})$



Unfold until exit (and then in reverse)

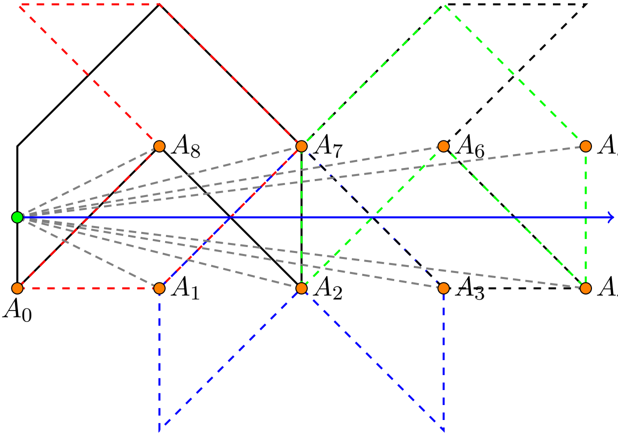


Divide and conquer

Define $\xi_m = \angle hA_m$ and require

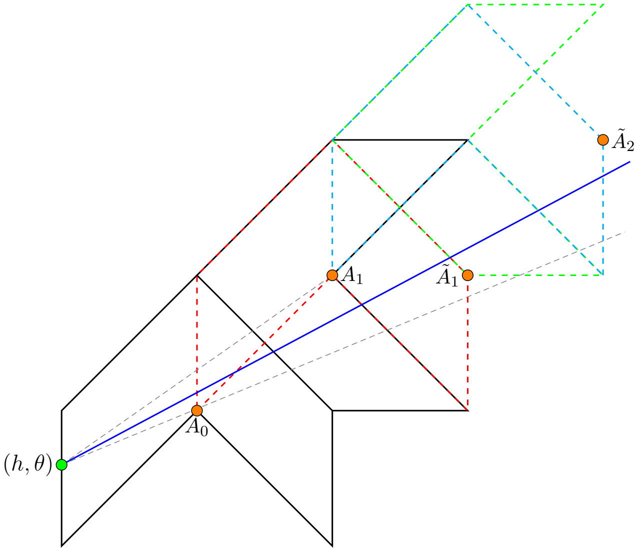
$$-\pi/2 = \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{N_d} = \pi/2$$

where $N_d < M_d$. For a first collision on Γ_0, we find

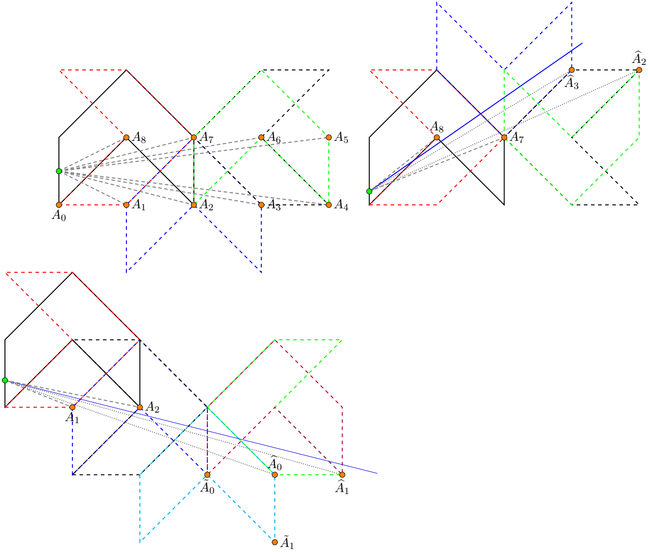


Nested singular directions

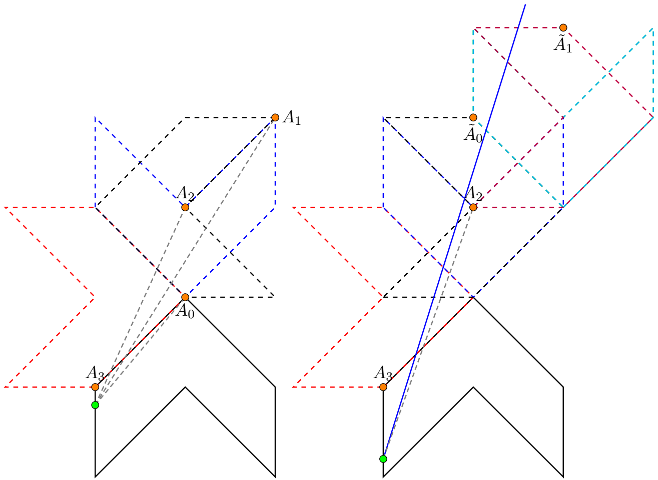
We have to prove that each $\widetilde{X}_j$ contains only regular trajectories.



A first bounce on Γ_0



A first bounce on Γ_3

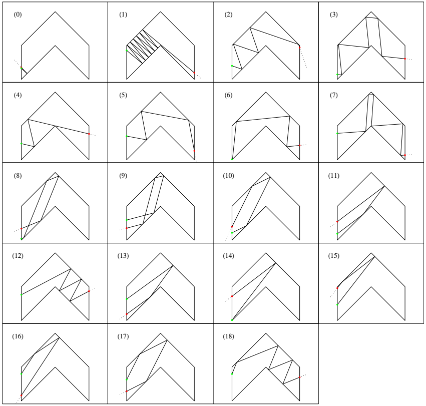


Behold

For $d = 1/2$ and $-\pi/2 < \theta < \pi/2$, we have

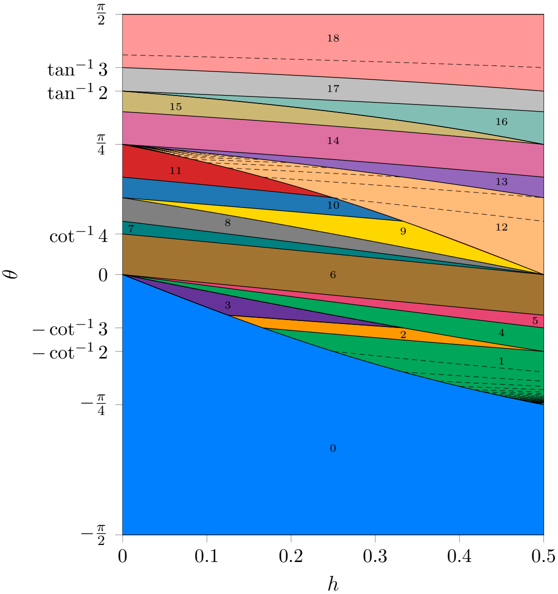
$$S(h,\theta) = \begin{cases} (-h\cot\theta, & -\theta + \frac{\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_0 \\ (h + \frac{1}{2}\left[\alpha(1+\tan\theta) + 3\tan\theta + 1\right], & +\theta) & \text{if } (h,\theta) \in \widetilde{X}_1 \cup \tilde{X}_4 \\ ((h+\frac{1}{2})\cot\theta + \frac{5}{2}, & -\theta - \frac{\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_2 \\ (h + \frac{1}{2}(1 + 5\tan\theta), & +\theta) & \text{if } (h,\theta) \in \widetilde{X}_3 \\ (h\cot\theta + 2d + 1, & -\theta - \frac{\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_5 \\ (h + (1+2d)\tan\theta, & +\theta) & \text{if } (h,\theta) \in \widetilde{X}_6 \\ (h + \frac{3}{2}\tan\theta - \frac{1}{2}, & +\theta) & \text{if } (h,\theta) \in \widetilde{X}_7 \\ (1 - (h + (1+2d)\tan\theta), & +\theta + \pi) & \text{if } (h,\theta) \in \widetilde{X}_8 \cup \widetilde{X}_9 \\ ((h-1)\cot\theta + 1 + 2d, & -\theta + \frac{3\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_{10} \\ (h + (1+d\kappa)\tan\theta - d\kappa, & +\theta) & \text{if } (h,\theta) \in \widetilde{X}_{12} \\ (-h + (d+1)(1-\tan\theta), & +\theta + \pi) & \text{if } (h,\theta) \in \widetilde{X}_{13} \\ (-h\cot\theta + (d+1)(\cot\theta - 1), & -\theta + \frac{3\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_{14} \\ (-h - \tan\theta + 2d + 1, & +\theta + \pi) & \text{if } (h,\theta) \in \widetilde{X}_{15} \\ (1 + 2d - (h + \tan\theta), & +\theta + \pi) & \text{if } (h,\theta) \in \widetilde{X}_{16} \\ (-\cot\theta(1 + 2d - h) + 1, & -\theta + \frac{3\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_{17} \\ (\cot\theta(-h + \frac{1}{2}(\gamma + \tan\theta(1-\gamma) + 3)), & -\theta + \frac{\pi}{2}) & \text{if } (h,\theta) \in \widetilde{X}_{18} \end{cases}$$

Image of the different trajectories



Itineraries
$\mathcal{L}\Gamma_0\mathcal{L}$
$\{\mathcal{L}\Gamma_0\Gamma_3\mathcal{R}, \mathcal{L}\Gamma_0\Gamma_3\Gamma_0\Gamma_3\mathcal{R}, \dots\}$
$\mathcal{L}\Gamma_0\Gamma_3\Gamma_0\Gamma_3\Gamma_2\mathcal{R}$
$\mathcal{L}\Gamma_0\Gamma_3\Gamma_0\Gamma_3\Gamma_2\Gamma_1\mathcal{R}$
$\mathcal{L}\Gamma_0\Gamma_3\Gamma_2\mathcal{R}$
$\mathcal{L}\Gamma_0\Gamma_3\Gamma_2\Gamma_1\mathcal{R}$
$\mathcal{L}\Gamma_0\Gamma_3\Gamma_2\Gamma_1\Gamma_2\Gamma_1\mathcal{R}$
$\mathcal{L}\Gamma_0\Gamma_3\Gamma_2\Gamma_0\mathcal{L}$
$\mathcal{L}\Gamma_0\Gamma_2\Gamma_3\Gamma_0\mathcal{L}$
$\mathcal{L}\Gamma_0\Gamma_2\Gamma_3\mathcal{L}$
$\mathcal{L}\Gamma_0\Gamma_2\mathcal{L}$
$\{\mathcal{L}\Gamma_2\Gamma_3\mathcal{R}, \mathcal{L}\Gamma_0\Gamma_3\Gamma_0\Gamma_3\mathcal{R}, \dots\}$
$\mathcal{L}\Gamma_2\Gamma_0\mathcal{L}$
$\mathcal{L}\Gamma_2\mathcal{L}$
$\mathcal{L}\Gamma_2\Gamma_3\mathcal{L}$
$\mathcal{L}\Gamma_3\Gamma_2\mathcal{L}$
$\mathcal{L}\Gamma_3\Gamma_2\Gamma_0\mathcal{L}$
$\{\mathcal{L}\Gamma_3\Gamma_2\Gamma_1\mathcal{R}, \mathcal{L}\Gamma_3\Gamma_2\Gamma_1\Gamma_2\Gamma_1\mathcal{R}\}$

The $\widetilde{X}_j$'s



The infinite horizon case

We get $d = 1$ for free (almost)

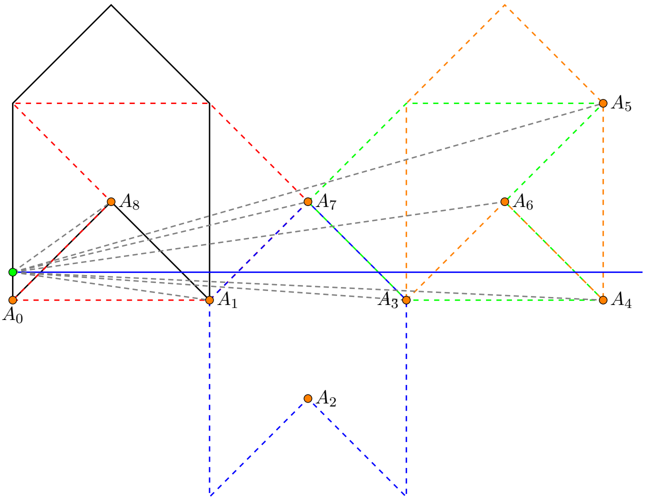
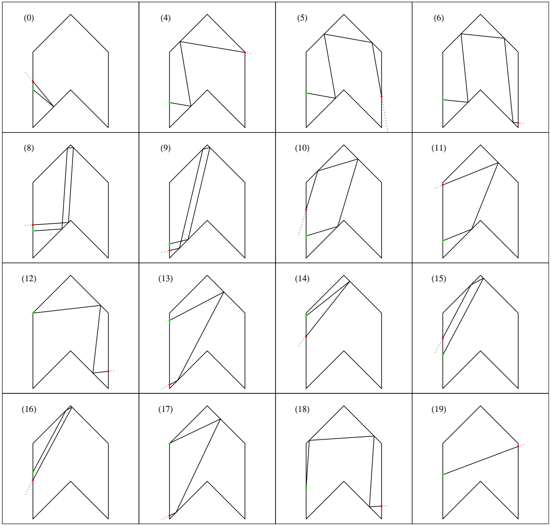
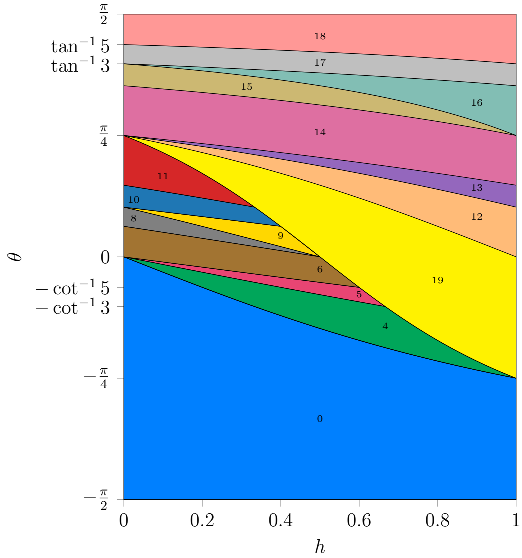


Image of the different trajectories



The different regions



Time and distance

From $\widehat{S}$ we get the particles' motion,

$$x_{n+1} = \begin{cases} x_n + 1 & \cos\theta_n \geq 0 \text{ and } \cos\theta_{n+1} \geq 0 \\ x_n - 1 & \cos\theta_n < 0 \text{ and } \cos\theta_{n+1} < 0 \\ x_n & \text{otherwise} \end{cases}$$

and crossing time,

$$\begin{aligned} \tau_0 &= -h\csc\theta & \tau_{10} &= (1-h)\csc\theta \\ \tau_1 &= d(\alpha+3)\sec\theta & \tau_{12} &= (1+\kappa d)\sec\theta \\ \tau_2 &= -(h+d)\csc\theta & \tau_{14} &= (d+1-h)\csc\theta \\ \tau_{3,7} &= (3/2+2d)\sec\theta & \tau_{15,16} &= \sec\theta \\ \tau_{4,11,13} &= (d+1)\sec\theta & \tau_{17} &= (1+2d)\csc\theta \\ \tau_5 &= -h\csc\theta & \tau_{18} &= (1+d(1+\gamma))\csc\theta \\ \tau_{6,8,9} &= (1+2d)\sec\theta & \tau_{19} &= \sec\theta \end{aligned}$$

Concluding remarks

- Computationally efficient
- Not limited to $\alpha = \pi/2$ and $d \in \{1/2, 1\}$ geometries!
- Maps accounting for all $d > 1$ have since been derived
- The map possessing the 'simplest' phase space for $\alpha = \pi/2$ is (almost certainly) obtained when $d = 1$

References I

Orchard J., Rondoni L., Mejía-Monasterio C., and Frascoli F. (2021), *Diffusion and escape from polygonal channels: extreme values and geometric effects.* J. Stat. Mech. **2021** (7), p. 073208.