On the Willmore Functional and Applications

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Some History

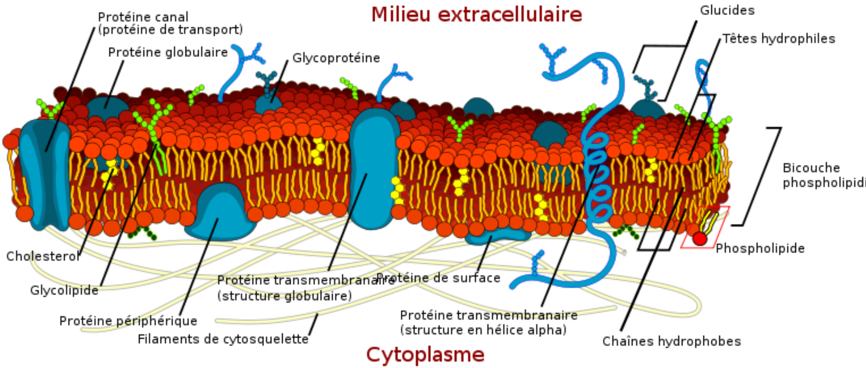
   

- ca. 1740: Leonhard Euler and Daniel Bernoulli study the 1-dim. elastica
- ca. 1815: Sophie Germain introduces **mean curvature**
- ca. 1925: Wilhelm Blaschke tries to tie **minimal surfaces** and **conformal invariance**
- in 1965: Tom Willmore (re)discovers the **Willmore functional**

A Biological Motivation

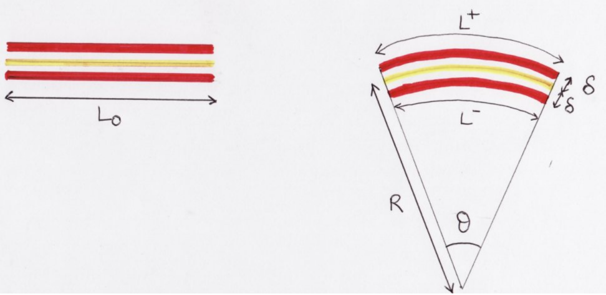
- Production rate: 2.4×10^6/ sec. Total: $25 \times 10^{12} = $ 70% of your cells
- No nucleus, no organelle, no DNA
- Size: $\sim 7\mu m$ diameter and $\sim 1\mu m$ thick.
- “Heatlhy" red blood cells look like biconcave disks
- Modelled as droplet of liquid (haemoglobin) within thin plasmic membrane

- Membrane is a lipid bilayer



- Membrane is incompressible (observation): area and volume constant
- If no shearing/stretching: which force dictates shape?
- Canham, Evans, Deuling, Helfrich postulate: the elastic energy.

Modeling the Energy



Exterior heads have available: $L^+ = (R + \delta)\theta$

Interior heads have available: $L^- = (R - \delta)\theta$

Tails have available: $L = R\theta \simeq L_0$

Energy modeled by elastic energy of a spring with intrinsic stiffness σ (N):

$$E^\pm = \frac{\sigma}{2L_0}(L^\pm - L_0)^2$$

Total energy of membrane element: $E = E^+ + E^- = \sigma\delta^2 \frac{L_0}{R^2}$

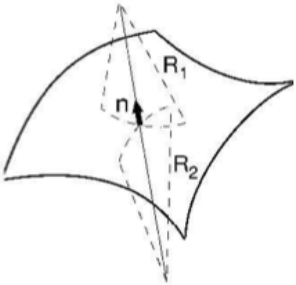
Sum all infinitesimal contributions to obtain the energy

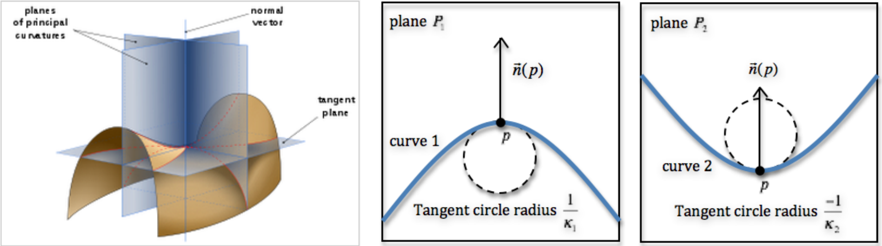
$$E = \mu \int_{\text{membrane}} \frac{1}{R^2} d\ell , \qquad \mu := \sigma\delta^2 .$$

R at point p is the radius of best-fitting circle tangent to membrane at p.

For a 2-dimensional membrane:

$$E = \mu \int_{\text{membrane}} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^2 ds \,, \qquad [\mu] = Joules \,.$$





Mean curvature : $H = \frac{1}{2}(\kappa_1 + \kappa_2)$

Willmore energy :

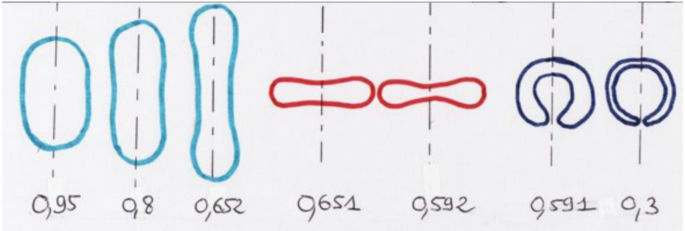
$$W(\Sigma) := \int_\Sigma H^2 dA$$

Minimizing the Energy

- Fixing area A and volume V, the energy is:

$$E(\Sigma) = \int_\Sigma H^2 dA - \alpha A(\Sigma) - \beta V(\Sigma) .$$

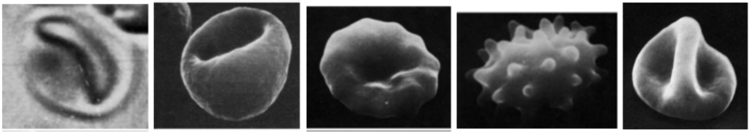
- Many shapes are possible (how many? what are they?)
- Define the reduced volume: $\lambda := 6\sqrt{\pi}\,\dfrac{V}{A^{3/2}}$.
- Imposing **axial symmetry**:



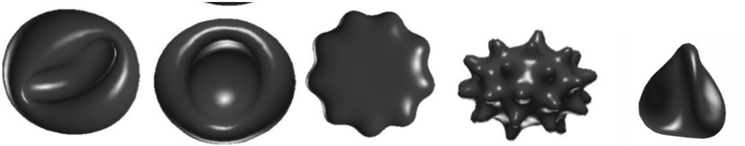
- Red blood cell: $V \simeq 100\mu m^3$ and $A \simeq 140\mu m^2$, so $\lambda \simeq 0.64$.

However...

- Not all red blood cells are biconcave disks

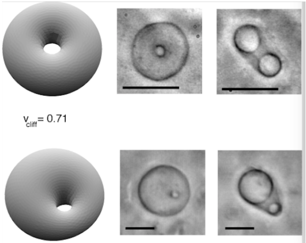


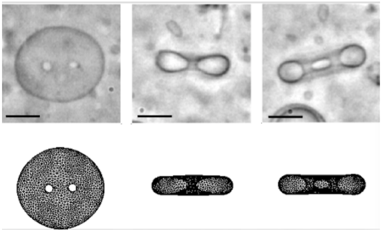
- Model needs to be modified to incorporate various physical effects



Vesicles

- Small tenuous objects with bilayer membrane of surfactant molecules (e.g. phospholipids). Look like deformed bags (1 to 100 μm).
- Useful to study emulsions and micro-emulsions, mixtures used in human activity (cooking, painting, oil extraction, cosmetics, drug delivery, ...)
- Unlike red blood cells, vesicles may display **non-trivial topology**.



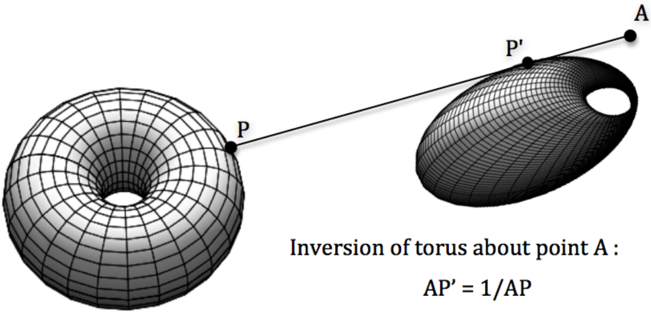


Introduction to the Willmore Functional

For a closed oriented surface Σ without boundary, the Willmore energy is

$$W(\Sigma) = \int_\Sigma H^2 \, dA \, .$$

Willmore energy invariant under reparametrizations in domain and conformal transformations (translation, rotation, dilation, inversion) in target



Willmore energy appears in various areas of science

- **cell biology** : bending energy of elastic membranes (biomembranes, vesicles, smectic A-liquid crystals, ...)
- **general relativity** : approximates Hawking mass
- **elasticity mechanics** : nonlinear plate theory. Γ-limit 3d$\rightarrow$2d
- **string theory**
- **optical design**
- **conformal geometry**

An Analogy

Let $\Gamma : [0,1] \to \mathbb{R}^3$ be a closed curve with curvature κ. Let the **total curvature**

$$w(\Gamma) \; := \; \int_0^1 \kappa(t)dt$$

Theorem [Fenchel, Borsuk]

$$w(\Gamma) \; \geq \; 2\pi$$

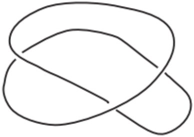
with equality if and only if curve is planar and convex.

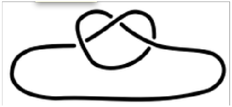
Theorem [Milnor, Fàry]

If

$$w(\Gamma) \; \leq \; 4\pi$$

then Γ is unknotted.





Theorem [Willmore, Pinkall]

Let Σ be an immersed closed surface. Then

$$W(\Sigma) \ \geq \ 4\pi$$

and equality holds if and only if Σ is round sphere.

Theorem [Li-Yau]

Let Σ be an immersed closed surface. If

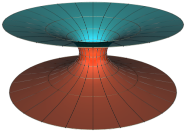
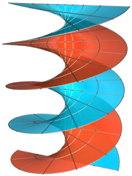
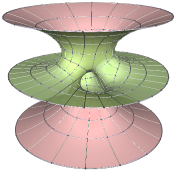
$$W(\Sigma) \ < \ 8\pi$$

then Σ is embedded.

Willmore Surfaces

Critical points of the Willmore energy are called Willmore surfaces.

- *minimal surfaces*: $H = 0$ with $W = 0$

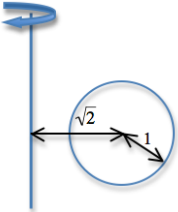
- CMC surfaces (i.e. $H = cst$) in general **not** Willmore
- composition of minimal immersion and conformal transformation
- The *round sphere* $\mathbb{S}^2$.

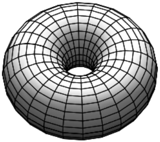
  Theorem [Bryant]

  A compact Willmore surface of genus zero in $\mathbb{R}^3$ is inversion of complete minimal surface with finite total curvature and embedded planar ends.

The Willmore torus T_0, projection onto $\mathbb{R}^3$ of Clifford torus :

$$(\theta, \varphi) \longmapsto \Big((\cos\varphi + \sqrt{2}) \cos\theta \,, (\cos\varphi + \sqrt{2}) \sin\theta \,, \sin\varphi \Big)$$

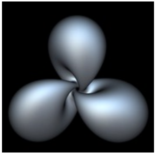


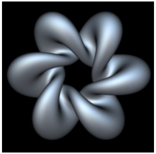


energy $W(T_0) = 2\pi^2$

Theorem [Marques-Neves]

Among all immersed tori in $\mathbb{R}^3$, the Willmore torus is the one which minimizes the Willmore energy (up to isometries of $\mathbb{S}^3$).





$$W(\Sigma) := \int_\Sigma H^2\, dA = \frac{1}{4}\int_\Sigma (\kappa_1+\kappa_2)^2 = \frac{1}{4}\int_\Sigma \left[(\kappa_1-\kappa_2)^2 + 4\kappa_1\kappa_2\right]$$

Gauss-Bonnet theorem : $\int_\Sigma \kappa_1\kappa_2\, dA = 2\pi\chi(\Sigma)$

$$\Longrightarrow \qquad W(\Sigma) = \frac{1}{4}\int_\Sigma (\kappa_1-\kappa_2)^2 dA + 2\pi\chi(\Sigma)$$

Willmore energy measures “sphericity"

Gauss equation : $4H^2 - 2\kappa_1\kappa_2 = |d\vec{n}|^2$

$$\Longrightarrow \qquad W(\Sigma) = \frac{1}{4}\int_\Sigma |d\vec{n}|^2\, dA + \pi\chi(\Sigma)$$

Willmore energy $\simeq$ Dirichlet energy of Gauss map $\vec{n}$

Also: $2H\vec{n} = \Delta\vec{\Phi}$ (negative Laplace-Beltrami for induced metric)

$$\Longrightarrow \qquad W(\Sigma) = \frac{1}{4}\int_\Sigma |\Delta\vec{\Phi}|^2 dA$$

Willmore energy $\simeq$ biharmonic energy of immersion $\vec{\Phi}$

The Willmore Equation

Theorem [Shadow, Thomsen, Weiner]

A smooth immersion $\vec{\Phi}$ is Willmore if and only if

$$\Delta_g H + 2H\,(H^2 - K) = 0\,,$$

$-\Delta_g$: Laplacian-Beltrami for induced metric.

What kind of questions might an analyst ask?

- meaning of solution (e.g. weak vs. strong)
- existence (e.g. with or without constraints, which constraints, ...)
- regularity, singularities, ...
- compactness
- ...

The Willmore Equation in Conservative Reformulation

Noether: an invariance in the energy gives rise to a conserved quantity.

Theorem [Rivière ('08), Y.B. ('14)]

- Translation: $\vec{T}^j := H\nabla^j\vec{n} - \vec{n}\nabla^j H + H^2\nabla^j\vec{\Phi}$ satisfies $\operatorname{div}\vec{T}^j = \vec{0}$.
- Rotation: $\operatorname{div}\big(\vec{T}^j \times \vec{\Phi} + H\vec{n}\times\nabla^j\vec{\Phi}\big) = \vec{0}$
- Dilation: $\operatorname{div}\big(\vec{T}^j \cdot \vec{\Phi}\big) = 0$

Last two divergence-free forms integrate to two potentials S and $\vec{R}$
An unexpected pleasant surprise:

$$\left\{\begin{array}{lcl} -|g|^{1/2}\Delta_g S & = & \partial_{x_1}\vec{n}\cdot\partial_{x_2}\vec{R} - \partial_{x_2}\vec{n}\cdot\partial_{x_1}\vec{R} \\ -|g|^{1/2}\Delta_g \vec{R} & = & \big(\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{R} - \partial_{x_2}\vec{n}\times\partial_{x_1}\vec{R}\big) + \big(\partial_{x_1}\vec{n}\,\partial_{x_2}S - \partial_{x_2}\vec{n}\,\partial_{x_1}S\big) \\ -|g|^{1/2}\Delta_g \vec{\Phi} & = & \big(\partial_{x_1}\vec{R}\times\partial_{x_2}\vec{\Phi} - \partial_{x_2}\vec{R}\times\partial_{x_1}\vec{\Phi}\big) + \big(\partial_{x_1}S\,\partial_{x_2}\vec{\Phi} - \partial_{x_2}S\,\partial_{x_1}\vec{\Phi}\big) \end{array}\right.$$

Theorem [Y.B., T. Rivière, Annals of Math. '14]

Σ arbitrary closed 2-dim manifold.
Modulo action of Möbius group of $\mathbb{R}^3$, the space of Willmore surfaces satisfying

$$W(\Sigma) \; < \; 8\pi - \delta$$

is compact in the C^ℓ topology, $\forall \, \ell \in \mathbb{N}, \, \delta > 0$.

Proof relies heavily on Noether's theorem applied to invariances of Willmore energy:

Willmore equation is truly a 2×2 system rather than a 4^{th}-order equation.

Open Questions

- Willmore conjecture in higher codimension?
- Analogue of the Willmore conjecture for genus ≥ 2.
- Understand the Seifert diagram rigorously
- Dynamical models
- Willmore-Plateau problem
- Replace the Willmore energy for surfaces by

$$\int_\Sigma |H|^p \qquad \text{for } p\text{-dimensional } \Sigma$$