Potts models with long-range interactions

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Outline:

1. Universality and range of interaction
2. Simulation of long-range models
3. The long-range Potts model

Contributors:

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1. Universality and range of interaction

Critical exponents: e.g.,

$$
\begin{aligned}
c &\propto |T_c - T|^{-\alpha} \, , \\
m_{\mathrm{spont}} &\propto |T_c - T|^{\beta} \, , \\
\chi &\propto |T_c - T|^{-\gamma} \, ,
\end{aligned}
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occur in *classes*.
Renormalization theory:
universality classes determined by

1. symmetry and nature of order parameter
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Is that all?

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No, mean-field models behave differently.

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1. symmetry and nature of order parameter
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3. range of interaction

Ising models with variable range:

$$\mathcal{H} = -\sum_{i,j} K(|\vec{r}_i - \vec{r}_j|) s_i s_j$$

where

$\mathcal{H}$ specifies energy
$i, j, \ldots$ label lattice sites
$K(r)$ is coupling for sites at distance r
$s_i = \pm 1$ are Ising variables

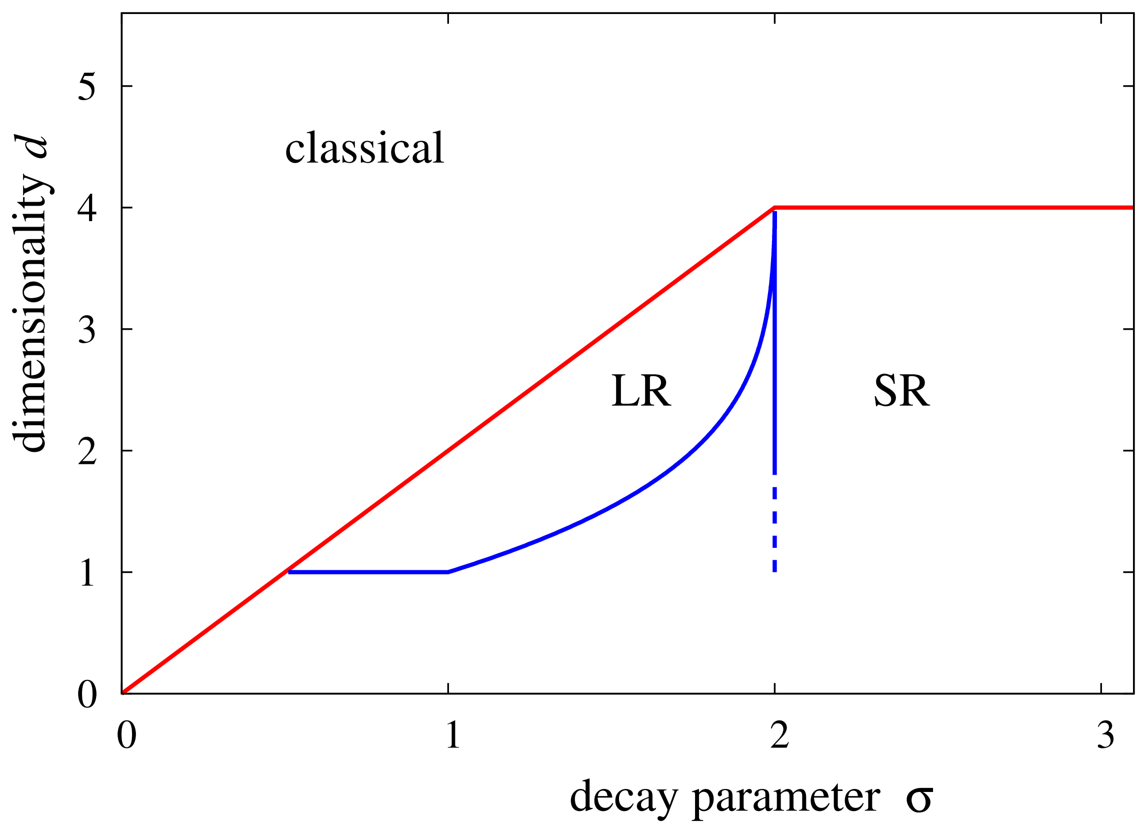
Two ways to vary range of interaction

1. $K(r) = K\theta(R - r)$ equivalent-neighbor model
2. $K(r) = Kr^{-(d+\sigma)}$ power-law decay with distance

Theories available!

Theory for models with power-law decay of interaction:

M.E. Fisher, S.-k. Ma and B.G. Nickel, Phys. Rev. Lett. 29, 917 (1972).



2. Simulation of long-range models

Accurate numerical tests of theory took over 20 years
Numerical verification by Metropolis method almost impossible
Time needed to generate independent configuration:

$$t_{\rm relax} = N_{\rm sw} N_{\rm sp} t_{\rm 1sp}$$

with

$N_{\rm sw} \simeq L^z$ critical slowing down, $z \approx 2$.
$N_{\rm sp} \simeq L^d$ # spins per sweep
$t_{\rm 1sp} \simeq L^d$ if every spin interacts with every other spin

$$t_{\rm relax} \simeq L^{2d+z} .$$

Cluster simulation reduces critical slowing down
Even larger reduction is possible!

Algorithm for FM Ising model with long-range interactions:

Cluster formation is percolation process
Probability to include an interacting neighbor s_j of spin s_i in cluster is

$$\delta_{s_i,s_j}(1 - e^{-K_{ij}})$$

Usual procedure: run over all neighbors s_j and check if

$$s_i = s_j \quad \text{AND} \quad R < 1 - e^{-K_{ij}}$$

For small K probability is small: procedure inefficient.

Better procedure: E. Luijten and H.B., Int. J. Mod. Phys. C6 359 (1995):
Neighbor selection in 2 rounds:

1. select candidate with probability $p_{i,j} \equiv 1 - e^{-K_{ij}}$
2. accept in cluster with probability δ_{s_i,s_j}

For step 1, transform probabilities as

$$P_{i,j} \equiv p_{i,j} \prod_{k=1}^{j-1} (1 - p_{i,k}) \,,$$

that s_j is the *first* neighbor that is selected by step 1.

Thus, if

$$\sum_{m=1}^{N} P_{i,m} > R$$

no neigbor spin is included, and if

$$\sum_{m=1}^{j-1} P_{i,m} < R < \sum_{m=1}^{j} P_{i,m} \,,$$

then, if $s_i = s_j$, s_j is included in the cluster.

If $j < N$, work remains.

The probability that s_l is the *next* neighbor that is selected by step 1, is

$$p_{i,l} \prod_{k=j+1}^{l-1} (1 - p_{i,k}) = \frac{P_{i,l}}{\prod_{k=1}^{j} (1 - p_{i,k})} \,,$$

Thus, draw another R and check for s_l with $j < l \leq N$:

$$\sum_{m=1}^{l-1} P_{i,m} < R \prod_{k=1}^{j} (1 - p_{i,k}) < \sum_{m=1}^{l} P_{i,m} \,,$$

This finds next neighbor s_l selected by step 1.

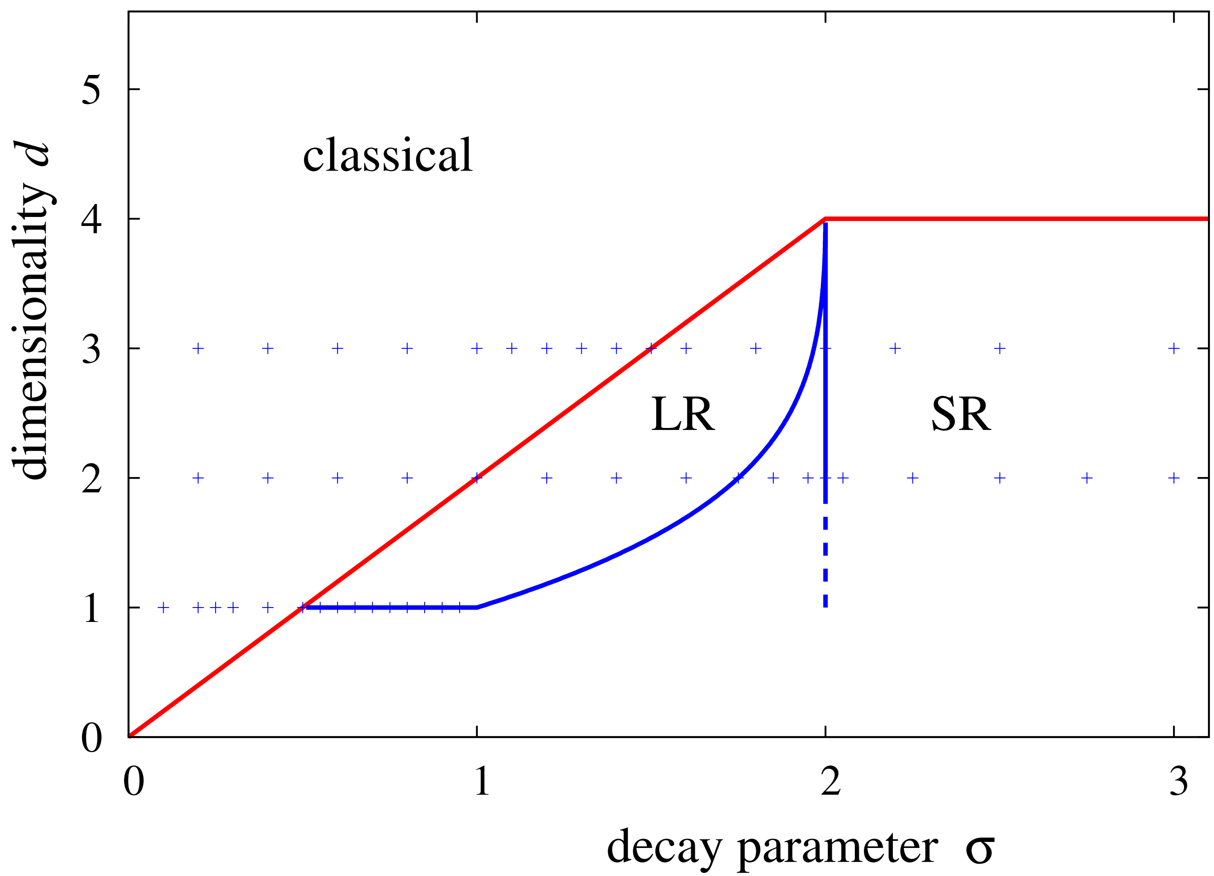
And so on.

Remarks:

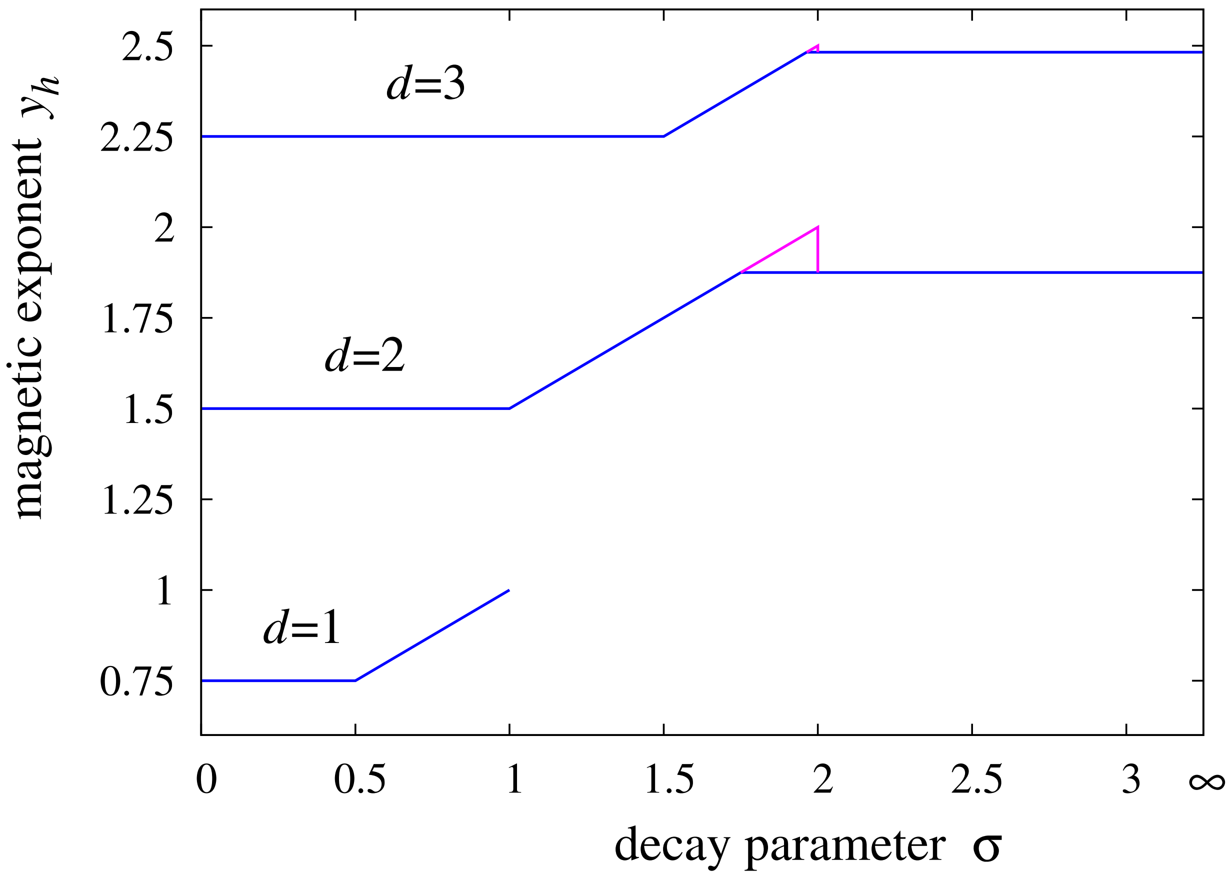
1. The $P_{i,j}$ can be rewritten $\tilde{P}(\vec{r}_i - \vec{r}_j)$, so can the partial sums: only N numbers.
2. Same for $\prod_{k=1}^{j}(1 - p_{i,k})$
3. These $2N$ numbers can be stored in a look-up table.

Result: time t_{relax} needed to generate indepent configuration
reduces roughly from order L^{2d+2} to order L^{d} !

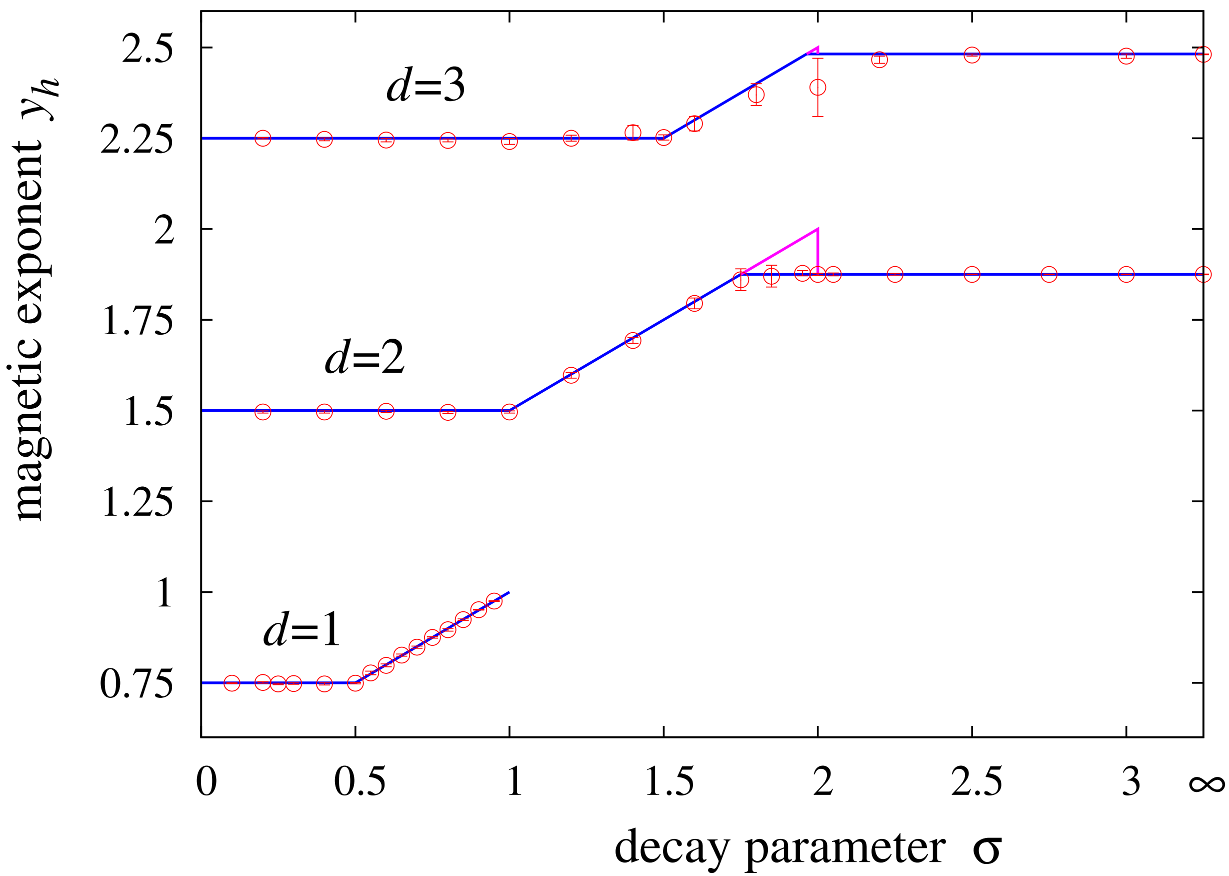
Calculations for following systems:



Theory due to:
Fisher, Ma and Nickel, Phys. Rev. Lett. 29, 917 (1972),
J. Sak, Phys. Rev. B 8, 281 (1973),
Gusmão and Theumann, Phys. Rev. B 28, 6545 (1983).



Comparison with Monte Carlo results:



3. The long-range Potts model

Equivalent-neighbor interactions in two dimensions:

$$\mathcal{H} = -\sum_{i,j} K(|\vec{r}_i - \vec{r}_j|)\delta_{s_i s_j}$$

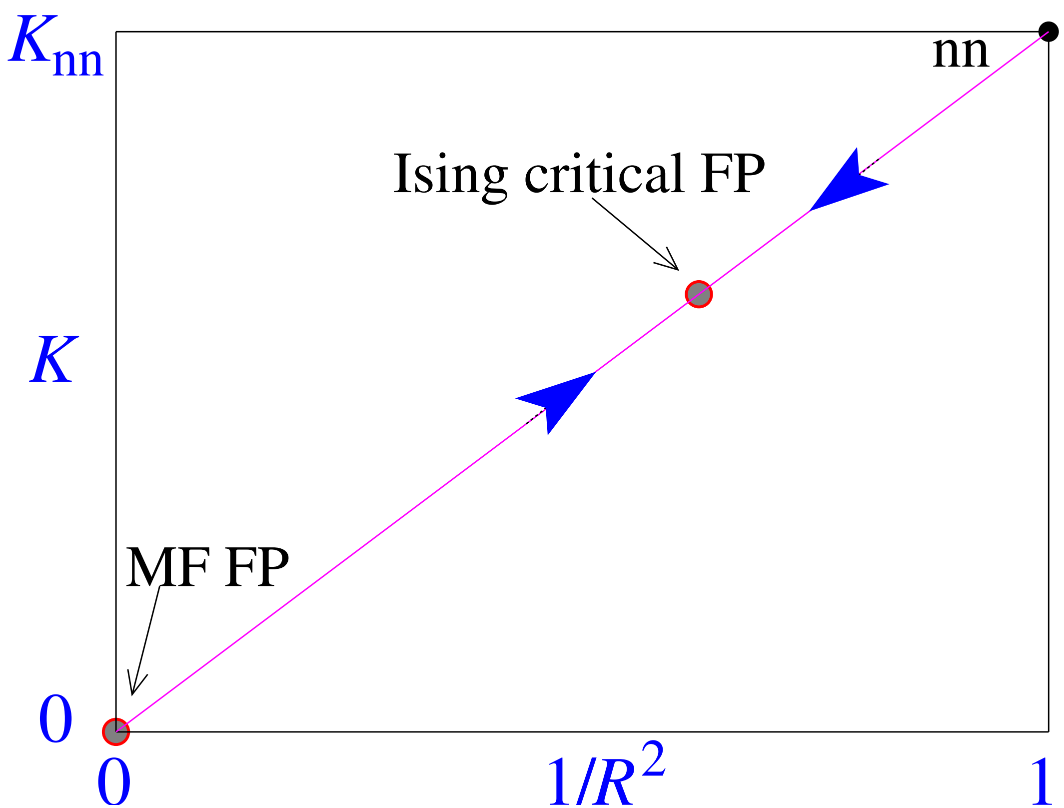
where

$s_k = 1,\ 2, \ldots, q$ are q-state Potts spins
$K(r) = K\theta(R-r)$: equivalent-neighbor model

For nn model, critical exponents depend “continuously” on q.

What happens when the range R increases?

$q = 2$: RG flow displays MF-Ising crossover



Mean-field limit $R \to \infty$ exactly solvable

1. $q < 2$: continuous transition, $\beta = 1$.
2. $q = 2$: continuous transition, $\beta = 2$.
3. $q > 2$: first-order transition.

For $R < \infty$, MC simulations:

Parameters	
R	range of interactions
q	number of Potts states
L	system size
K	strength of couplings

Sampled quantities	
ρ_i	density of variables in state i
n_c	number of clusters in configuration
c_j	size of j-th cluster

Squared magnetization follows as

$$\langle m^2 \rangle = \frac{1}{q-1} \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \langle (\rho_i - \rho_j)^2 \rangle$$

which can, for integers $q > 1$, be expressed in terms of cluster sizes:

$$\langle m^2 \rangle = \left\langle \sum_{i=1}^{n_c} c_i^2 \right\rangle$$

and fourth moment of m is

$$\langle m^4 \rangle = \frac{q+1}{q-1} \left\langle \left(\sum_{i=1}^{n_c} c_i^2 \right)^2 \right\rangle - \frac{2}{q-1} \left\langle \sum_{i=1}^{n_c} c_i^4 \right\rangle$$

Then form Binder ratio

$$Q \equiv \frac{\langle m^2 \rangle^2}{\langle m^2 \rangle^4}$$

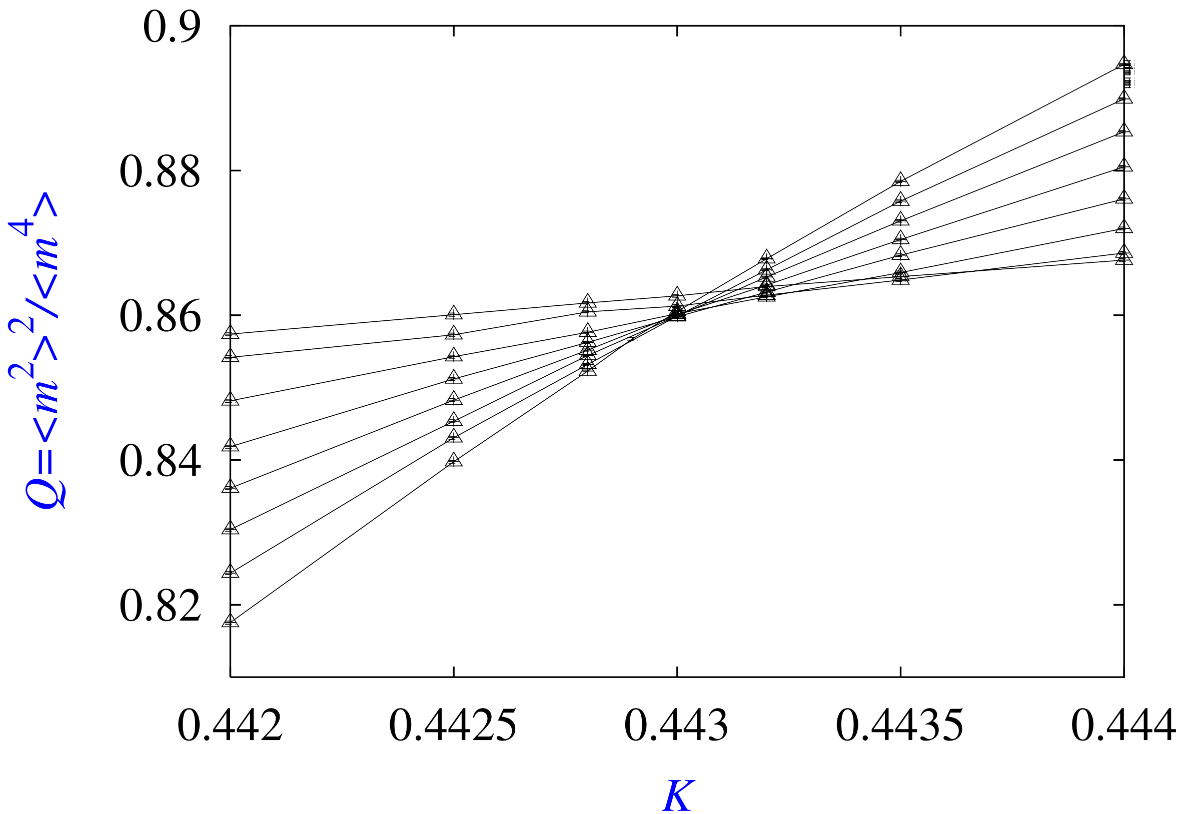
which, near a critical point $K = K_c$, behaves as

$$Q \simeq Q_c + a_1 (K - K_c) L^{y_t} + b L^{y_i} + \cdots$$

where

Q_c	is a universal constant
$y_t > 0$	is the temperature exponent
$y_i < 0$	is the irrelevant exponent

Example for $q = 3$, $R = \sqrt{2}$ ($z = 8$ equivalent neighbors)



system sizes $L = 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24,\ 30,\ 36,\ 42$ and 48.

Differentiation of scaling formula for Q:

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}K}\right|_{K=K_c} = L^{y_t}\left(a_1 + cL^{y_i} + \cdots\right),$$

where a_1 is the leading amplitude.

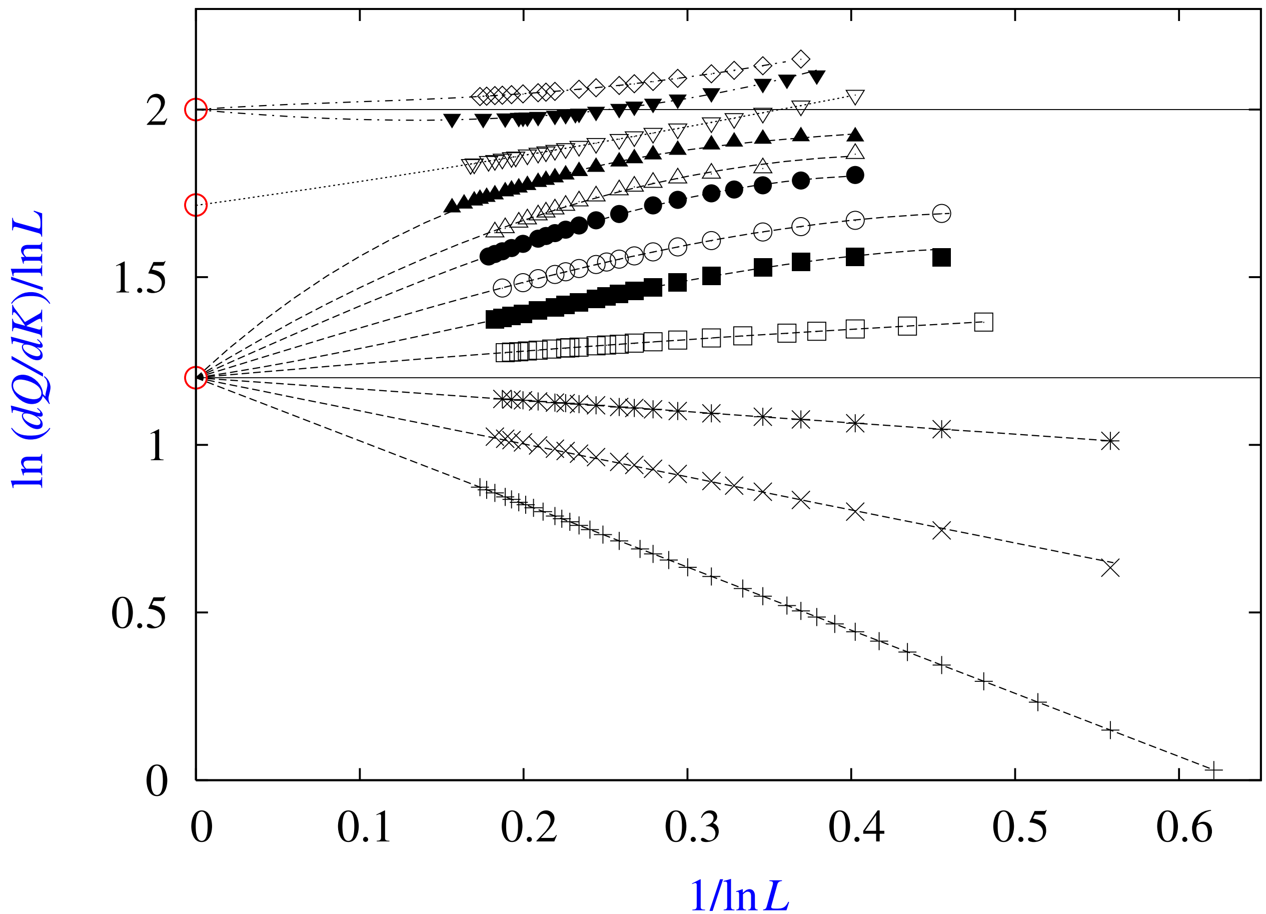
Other contributions relatively unimportant for $L \to \infty$.

Data analysis uses quantity

$$\frac{\ln(\mathrm{d}Q/\mathrm{d}K)}{\ln L} = y_t + \frac{\ln a_1 + (c/a_1)L^{y_i} + \cdots}{\ln L}$$

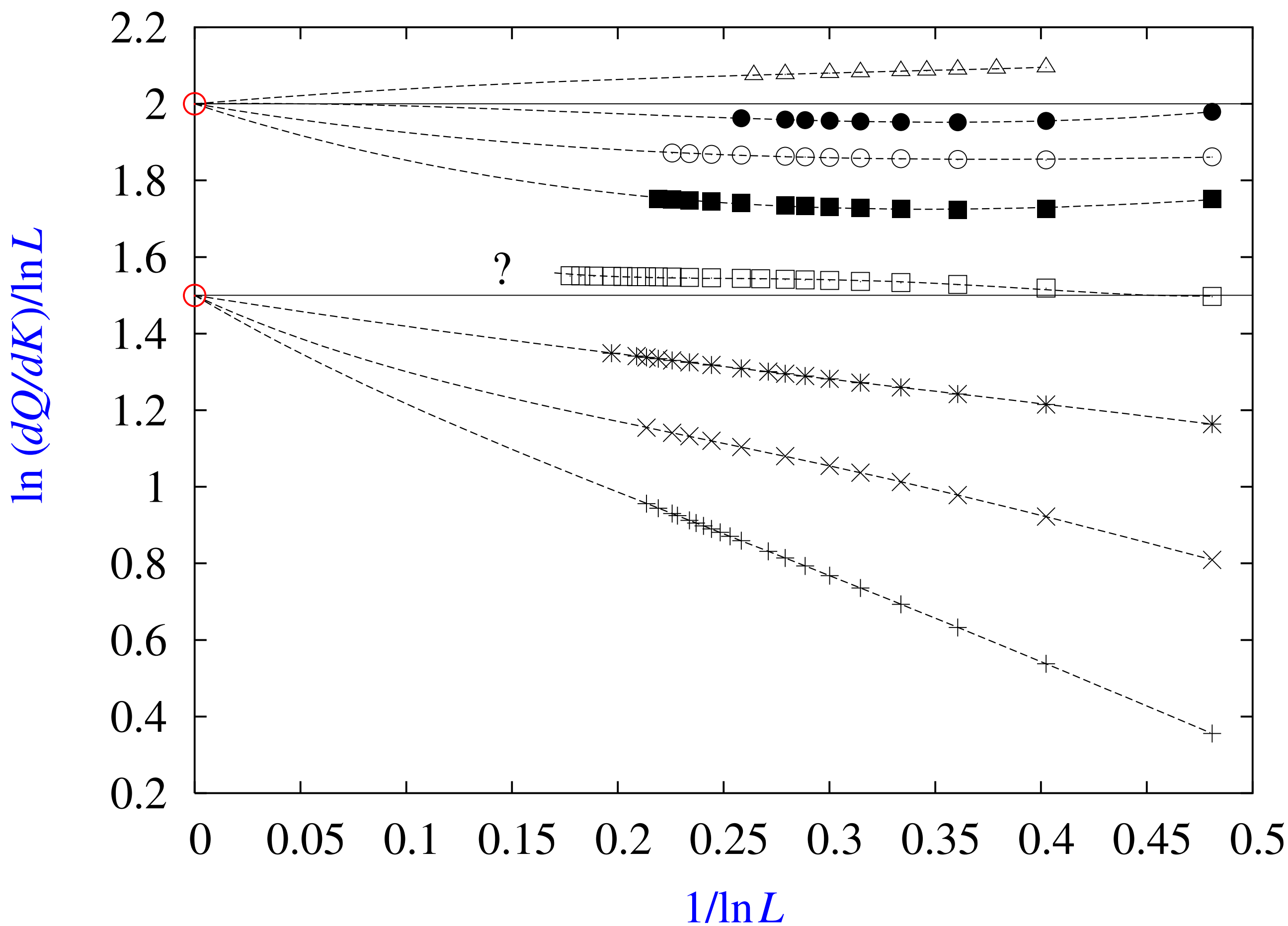
which will reveal y_t for sufficiently large L.

$q = 3$:



Results (from bottom to top) for $z = 4,\ 8,\ 12,\ 20,\ 28,\ 36,\ 48,\ 56,\ 68,\ 80,\ 100$ and 120.

$q = 4$:



Results (from bottom to top) for $z = 4,\ 8,\ 12,\ 20,\ 28,\ 36,\ 44$ and 60.

Interpretation for $q = 3$:

For $z \lesssim 80$ convergence to $y_t = 6/5$ for critical $q = 3$ Potts model;
for $z \approx 80$ convergence to $y_t = 12/7$ for tricritical $q = 3$ Potts model;
for $z \gtrsim 80$ convergence to $y_t = 2$ for first-order transition.

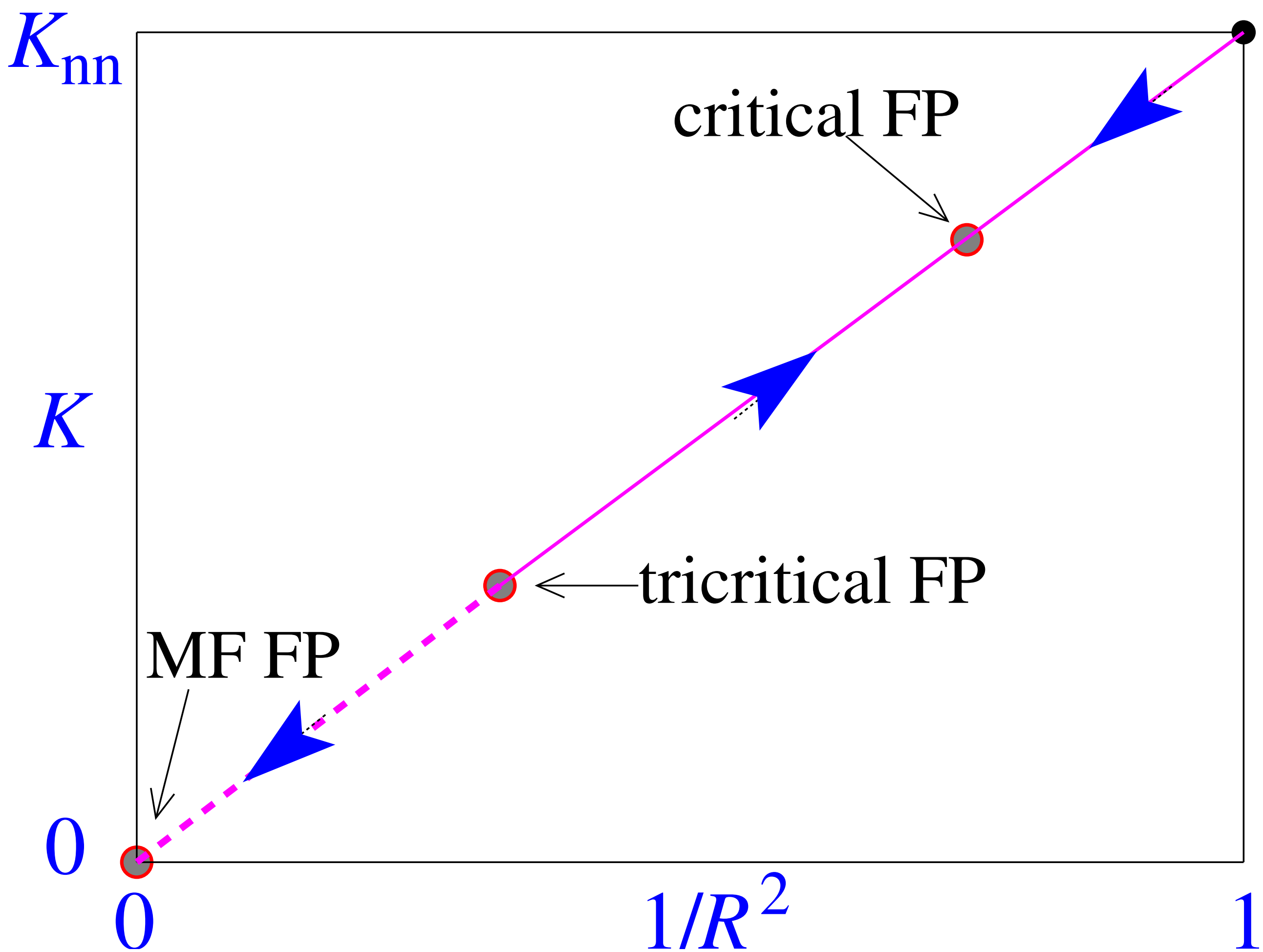
Interpretation for $q = 4$:

For $z \lesssim 20$ convergence to $y_t = 3/2$ for critical $q = 4$ Potts model;
for $z \gtrsim 20$ convergence to $y_t = 2$ for first-order transition.

Conclusion:

MF to short-range crossover essentially different from Ising case.

RG and phase diagram for $q = 3$



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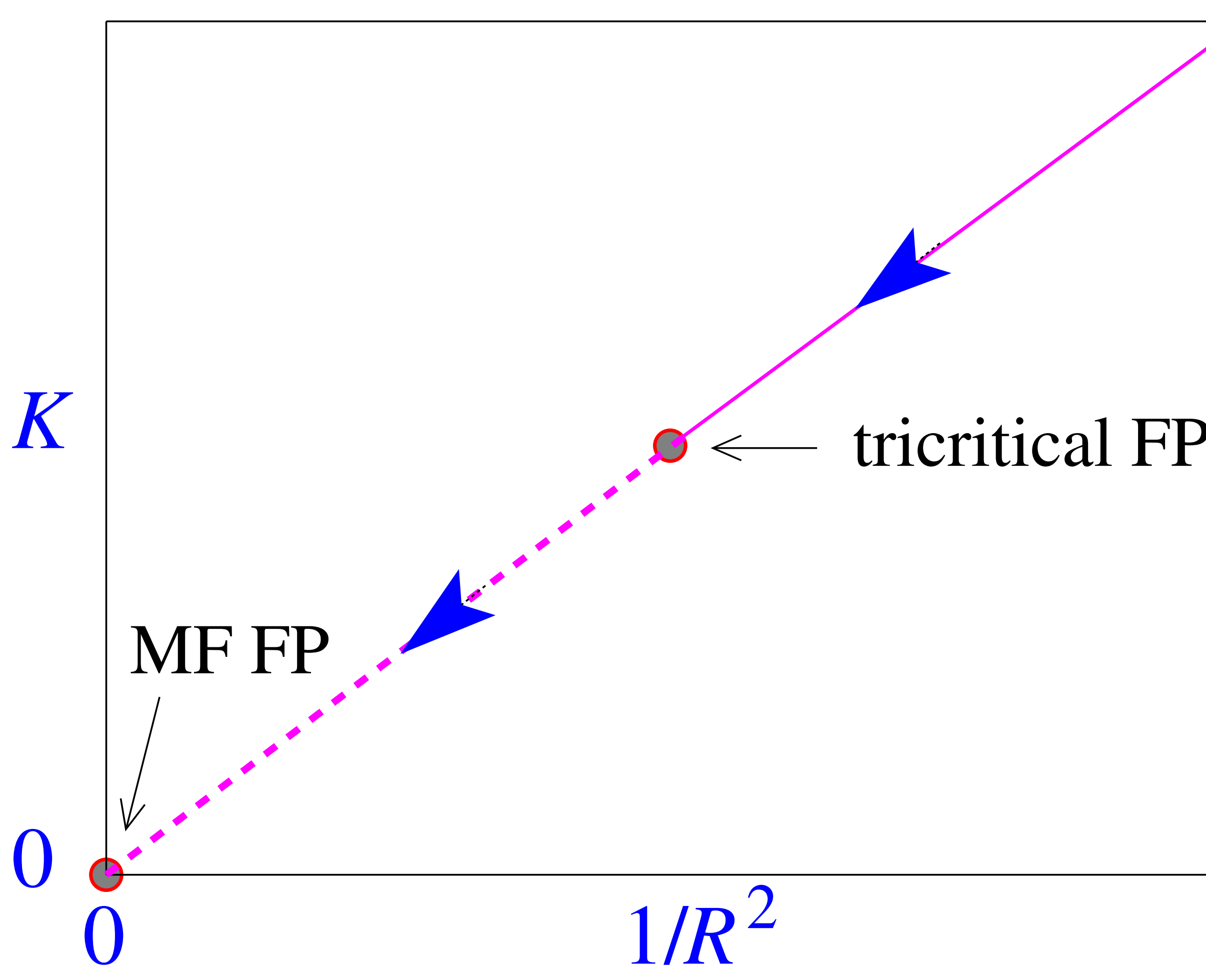

Conclusions

- Efficient simulations of long-range models

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- Tricriticality and first order transitions in $2 < q \leq 4$ Potts models

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- Tricriticality and first order transitions in $2 < q \leq 4$ Potts models
- Applications to neural networks, spin glasses
- Applications to dipolar and ionic systems

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